ON A CONJECTURE OF ZHAO RELATED TO STANDARD RELATIONS AMONG CYCLOTOMIC MULTIPLE ZETA VALUES

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Abstract. We provide a proof of a conjecture by Zhao concerning the structure of certain relations among cyclotomic multiple zeta values in weight two. We formulate this conjecture in a broader algebraic setting in which we give a natural equivalence between two schemes attached to a finite abelian group G. In particular, when G is the group of roots of unity, these schemes describe the standard relations among cyclotomic multiple zeta values.

Contents

Introduction

The purpose of this paper is to give a proof of a conjecture of Zhao and to compare two distinct frameworks describing the algebraic relations among multiple zeta values and more generally multiple polylogarithms at roots of unity. *Multiple polylogarithms* generalize the classical polylogarithms and are defined for $k_1, \dots, k_r \geq 1$ by

$$\mathrm{Li}_{(k_1,\dots,k_r)}(z_1, \dots, z_r) = \sum_{n_1 > \cdots > n_r > 0} \frac{z_1^{n_1} \cdots z_r^{n_r}}{n_1^{k_1} \cdots n_r^{k_r}}, \tag{1}$$

which converge for suitable values of z_i. In (1) we call r the *depth* and $k_1 + \cdots + k_r$ the *weight*. The special case where $k_1 \geq 2$ and $z_1 = \cdots = z_r = 1$, yields *multiple zeta values (MZVs)* $\zeta(k_1, \dots, k_r) = \mathrm{Li}_{(k_1,\dots,k_r)}(1, \dots, 1)$. From the perspective of Ihara, Kaneko, and Zagier [IKZ06], conjecturally all algebraic relations of MZVs are a consequence of the extended double shuffle (EDS) relations. Another special case for the values of z_i in (1) is when they are N-th roots of unity. The resulting values are sometimes referred to as colored or *cyclotomic multiple zeta values* of level N. Besides the extended double shuffle relations these values also satisfy the so-called finite and regularized distribution relations. The main result of this work is the following.

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Theorem I ([Zha10, Conjecture 4.7])**.** *In weight two, all regularized distribution relations are consequences of the extended double shuffle relations, finite distribution relations of weight one and finite distribution relations of depth two.*

We will prove a more general variant of Theorem I, which provides a similar statement regarding a generalization of the distribution and double shuffle relations. This generalization will be presented within two distinct algebraic frameworks. Theorem I will follow as a consequence of Theorem 2.10 and Proposition 2.11, established in this broader context. One framework extends the algebraic setup of Ihara, Kaneko, and Zagier ([IKZ06]), while the other was introduced by Racinet ([Rac02]). In the following, we will outline these two frameworks and compare them.

Following Ihara, Kaneko, and Zagier ([IKZ06]), the double shuffle relations can be viewed as a comparison between two product formulas satisfied by MZVs. To describe them, one introduces a space $\mathfrak{H}^0$ equipped with two products: ш (shuffle product) and $*$ (harmonic product). MZVs are then seen as an algebra homomorphism from $\mathfrak{H}^0$ to $\mathbb{R}$ with respect to both products. By extending this map in two natural ways, through regularization, to two algebra homomorphisms from a larger algebra $\mathfrak{H}^0 \subset \mathfrak{H}^1$ to $\mathbb{R}[T]$, along with an explicit comparison map $\rho : \mathbb{R}[T] \to \mathbb{R}[T]$ between both regularizations, the extended double shuffle relations are established. In the context of cyclotomic multiple zeta values, one can similarly describe the algebraic relations satisfied by these values (e.g. [AK04, Tas21, YZ16]) by considering them as algebra homomorphisms of an extension of $\mathfrak{H}^0$.

In this paper, we generalize this algebraic framework by introducing a space $\mathfrak{H}^0_G$ associated with a finite abelian group G. We then provide a definition of what it means for a $\mathbb{Q}$-linear map $Z_{\mathbf{A}}$ from $\mathfrak{H}^0_G$ to a $\mathbb{Q}$-algebra $\mathbf{A}$ to satisfy the extended double shuffle relations. In the cases where G is trivial or where G is the group of N-th roots of unity, this notion reduces to the classical cases of multiple zeta values and their level N analogues, respectively. For a given group G and $\mathbb{Q}$-algebra $\mathbf{A}$ we then introduce the set $\mathsf{EDS}(G)(A)$ of all such maps.

A somehow dual point of view of the above story is Racinet's ([Rac02]) approach of the study of MZVs through the lens of groupoid-enriched Hopf algebras, emphasizing the role of group actions and the double shuffle relations in this context. In his framework, one considers formal non-commutative power series with coefficients in $\mathbf{A}$, which are, after a certain correction, group-like for two different coproducts. Denote by $\mathsf{DMR}(G)(\mathbf{A})$ (Definition 1.14) the set of all such series. This formalism builds upon the notion of Drinfeld associator introduced in [Dri91] which is a generating series of MZVs that satisfies associator relations describing periods of mixed Tate motives. It is shown in [Fur11] that associator relations imply double shuffle relations and it is conjectured to be equivalent. A cyclotomic analogue of Drinfeld associator called cyclotomic associator is studied by [Enr08] and its connection with double shuffle relations was established in [Fur13].

Both the setup introduced by Ihara, Kaneko, and Zagier and the one introduced by Racinet have been used by various authors to describe the relations among multiple polylogarithms or (cyclotomic) multiple zeta values. Their equivalence is in some sense common knowledge among the community but, as far as the authors know, it was not written down precisely somewhere so far, except for the classical case of trivial G (cf. [ENR03, Proposition 3.1], [Bur23]). In this work, we give an explicit comparison for arbitrary G by assigning to a $Z_{\mathbf{A}} \in \mathsf{EDS}(G)(\mathbf{A})$ an element $\Phi_{Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{ш}} \in \mathsf{DMR}(G)(\mathbf{A})$ (Definition 1.16), such that we have the following

Theorem II (Theorem 1.18)**.** *The following map is bijective*

$$\begin{array}{ccc} \mathsf{EDS}(G)(\mathbf{A}) & \longrightarrow & \mathsf{DMR}(G)(\mathbf{A}), \\ Z_{\mathbf{A}} & \longmapsto & \Phi_{Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{ш}}. \end{array} \tag{2}$$

Racinet's viewpoint introduces a more general setup then the one of Ihara, Kaneko, and Zagier, accommodating also the so-called distribution relations. For multiple polylogarithms these were explicited by Goncharov [Gon98, Gon01].

Distribution relations are not a consequence of the extended double shuffle relations and they are trivial in the classical case if G is trivial. Both are part of the so-called *standard relations* among the cyclotomic multiple zeta values ([Zha10]) in the case if G is the group of roots of unity. The distribution relations were incorporated by Racinet by introducing a subset $\mathsf{DMRD}(G)(\mathbf{A}) \subset \mathsf{DMR}(G)(\mathbf{A})$, which gives the subset of elements that additionally satisfy these relations. As a counterpart to this, we introduce the subset $\mathsf{EDSD}(G)(\mathbf{A}) \subset \mathsf{EDS}(G)(\mathbf{A})$ (Definition 2.8) and show the following.

Theorem III (Theorem 2.9)**.** *The bijection* (2) *restricts to a bijection*

$$\mathsf{EDSD}(G)(\mathbf{A}) \longrightarrow \mathsf{DMRD}(G)(\mathbf{A}).$$

While the result of Theorem II may be regarded as a well-known fact among experts, it has not yet been explicitly documented in the literature in the general form presented here. Nevertheless, we emphasize that the result in Theorem III is novel, as the concept of the space $\mathsf{EDSD}(G)(\mathbf{A})$ is entirely new and has not been discussed or introduced elsewhere.

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Notation. Throughout this paper, we will use the following notations

(i) G is a finite abelian (multiplicative) group.
(ii) $X_G := \{x_0\} \sqcup \{x_g | g \in G\}$.
(iii) $Y_G := \{y_{n,g} | (n, g) \in \mathbb{Z}_{>0} \times G\}$.
(iv) $\mathbf{A}$ is a commutative $\mathbb{Q}$-algebra with unit.
(v) $\mathcal{L}^*$ is the monoid generated by a set $\mathcal{L}$.

1. Double shuffle relations

In this section, we review two fundamental formalisms used in the algebraic setup of double shuffle relations. We recall Racinet's formalism as presented in [Rac02] encoding double shuffle relations into a scheme $\mathsf{DMR}(G)$ attached to a group G. Another formalism was developed by Ihara, Kaneko and Zagier [IKZ06] for trivial G and Arakawa, Kaneko [AK04] for general G. Following Racinet's approach, we encode this formalism into a scheme denoted as $\mathsf{EDS}(G)$. The section concludes by proving that the schemes $\mathsf{DMR}(G)$ and $\mathsf{EDS}(G)$ are isomorphic, thus establishing formally the equivalence between the two formalisms.

1.1. The scheme EDS of extended double shuffle relations.

Denote by $\mathfrak{H}_G := \mathbb{Q}\langle X_G \rangle$ the free non-commutative associative polynomial $\mathbb{Q}$-algebra with unit over the alphabet X_G. Set

$$\mathfrak{H}_G^1 := \mathbb{Q} + \sum_{g\in G} \mathfrak{H}_G x_g \quad \text{and} \quad \mathfrak{H}_G^0 := \mathbb{Q} + \sum_{g\in G} x_0 \mathfrak{H}_G x_g + \sum_{g\in G\neq\{1\}} x_g \mathfrak{H}_G^1.$$

One checks that we have the following inclusion of $\mathbb{Q}$-algebras

$$\mathfrak{H}_G^0 \subset \mathfrak{H}_G^1 \subset \mathfrak{H}_G.$$

For $(n, g) \in \mathbb{Z}_{>0} \times G$, the assignment $y_{n,g} \mapsto x_0^{n-1} x_g$ establishes a $\mathbb{Q}$-algebra isomorphism $\mathbb{Q}\langle Y_G \rangle \simeq \mathfrak{H}_G^1$. We shall identify these two algebras thanks to this isomorphism.

Definition 1.1. We define the *shuffle product* ш on $\mathfrak{H}_G$ inductively by

$$\begin{cases} 1 \sqcup\!\sqcup w = w \sqcup\!\sqcup 1 = w & \text{for } w \text{ a word in } \mathfrak{H}_G \\ uw_1 \sqcup\!\sqcup vw_2 = u(w_1 \sqcup\!\sqcup vw_2) + v(uw_1 \sqcup\!\sqcup w_2) & \text{for } w_1, w_2 \text{ words in } \mathfrak{H}_G \text{ and } u, v \in X_G, \end{cases}$$

and then extending by $\mathbb{Q}$-bilinearity.

The product ш gives $\mathfrak{H}_G$ the structure of a commutative $\mathbb{Q}$-algebra, which we denote by $\mathfrak{H}_{G,\sqcup\!\sqcup}$. The subspaces $\mathfrak{H}_G^1$ and $\mathfrak{H}_G^0$ then become subalgebras of $\mathfrak{H}_{G,\sqcup\!\sqcup}$ denoted $\mathfrak{H}_{G,\sqcup\!\sqcup}^1$ and $\mathfrak{H}_{G,\sqcup\!\sqcup}^0$ respectively.

Definition 1.2. We define the *harmonic product* $*$ on $\mathfrak{H}_G^1$ inductively by

$$\begin{cases} 1 * w = w * 1 = w \\ y_{n_1,g_1} w_1 * y_{n_2,g_2} w_2 = y_{n_1,g_1}(w_1 * y_{n_2,g_2} w_2) + y_{n_2,g_2}(y_{n_1,g_1} w_1 * w_2) + y_{n_1+n_2,\ g_1 g_2}(w_1 * w_2), \end{cases}$$

for words w, w_1, w_2 in $\mathfrak{H}_G^1$, $(n_1, g_1), (n_2, g_2) \in \mathbb{Z}_{>0} \times G$; and then extending by $\mathbb{Q}$-bilinearity.

One checks that $\mathfrak{H}_{G,*}^1 := \left(\mathfrak{H}_G^1, *\right)$ is a commutative $\mathbb{Q}$-algebra and that $\mathfrak{H}_{G,*}^0 := \left(\mathfrak{H}_G^0, *\right)$ is a subalgebra of $\mathfrak{H}_{G,*}^1$.

Let q_G be the $\mathbb{Q}$-linear automorphism of $\mathfrak{H}_G$ given by ([Rac02, §2.2.7])

$$\mathrm{q}_G(x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_2} \cdots x_0^{n_r-1} x_{g_r} x_0^{n_{r+1}-1}) = x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_2 g_1^{-1}} \cdots x_0^{n_r-1} x_{g_r g_{r-1}^{-1}} x_0^{n_{r+1}-1},$$

for $r, n_1, \dots, n_{r+1} \in \mathbb{Z}_{>0}$ and $g_1, \dots, g_r \in G$. Its reciprocal is given by

$$x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_2} \cdots x_0^{n_r-1} x_{g_r} x_0^{n_{r+1}-1} \mapsto x_0^{n_1-1} x_{g_1} x_0^{n_2-1} x_{g_1 g_2} \cdots x_0^{n_r-1} x_{g_1 \cdots g_r} x_0^{n_{r+1}-1}.$$

One checks that q_G restricts to a $\mathbb{Q}$-linear automorphism of $\mathfrak{H}_G^1$ which we denote q_G as well and is given by

$$\mathrm{q}_G(y_{n_1,g_1} y_{n_2,g_2} \cdots y_{n_r,g_r}) = y_{n_1,g_1} y_{n_2,g_2 g_1^{-1}} \cdots y_{n_r, g_r g_{r-1}^{-1}}.$$

Moreover, q_G also restricts to an $\mathbf{A}$-module automorphism of $\mathfrak{H}_G^0$ also denoted q_G.

Definition 1.3. A $\mathbb{Q}$-linear map $Z_{\mathbf{A}} : \mathfrak{H}^0 \to \mathbf{A}$ is said to have the *finite double shuffle* property if it is an algebra homomorphism for ш and if $Z_{\mathbf{A}} \circ \mathrm{q}_G^{-1}$ is an algebra homomorphism for $*$. i.e.

$$Z_{\mathbf{A}}(w_1) Z_{\mathbf{A}}(w_2) = Z_{\mathbf{A}}(w_1 \sqcup\!\sqcup w_2),$$
$$Z_{\mathbf{A}}(\mathrm{q}_G^{-1}(w_1)) Z_{\mathbf{A}}(\mathrm{q}_G^{-1}(w_2)) = Z_{\mathbf{A}}(\mathrm{q}_G^{-1}(w_1 * w_2)),$$

for all $w_1, w_2 \in \mathfrak{H}^0$.

Example 1.4. In the classical case ([AK04],[IKZ06]), one takes $\mathbf{A} = \mathbb{C}$ and for a fixed $N \geq 1$ one considers the group of N-roots of unity $G = \mu_N$. Let us consider the $\mathbb{Q}$-linear map $Z_{\mathbb{C}} : \mathfrak{H}_{\mu_N}^0 \to \mathbb{C}$ given by $1 \mapsto 1$ and for $k > 0$,

$$u_1 \cdots u_k \longmapsto \int_0^1 \Omega_{u_1} \cdots \Omega_{u_k},$$

where $\Omega_{x_0} = \frac{dt}{t}$ and $\Omega_{x_z} = \frac{dt}{z^{-1} - t}$. Here, the iterated integrals are recursively defined by

$$\begin{cases} \displaystyle\int_a^b \omega_1 = \int_a^b f_1(t) dt \\ \displaystyle\int_a^b \omega_1 \cdots \omega_n = \int_a^b \left(\int_a^t \omega_2 \cdots \omega_n \right) f_1(t) dt & \text{if } n > 1, \end{cases}$$

where $\omega_1, \ldots, \omega_n$ are complex valued differential 1-forms defined on a real interval $[a, b]$ such that $\omega_i = f_i(t)dt$ with complex functions $f_1, \ldots, f_n$. It is known that (see, for example, [Gon98, Theorem 2.1])

$$\begin{aligned}
\mathrm{Li}_{(k_1,\ldots,k_r)}(z_1, \ldots, z_r) &= \int_0^1 \Omega_{x_0}^{k_1-1}\Omega_{x_{z_1}}\Omega_{x_0}^{k_2-1}\Omega_{x_{z_1 z_2}} \cdots \Omega_{x_0}^{k_r-1}\Omega_{x_{z_1\cdots z_r}} \\
&= Z_{\mathbb{C}} \circ \mathrm{q}_{\mu_N}^{-1}(x_0^{k_1-1}x_{z_1}x_0^{k_2-1}x_{z_2}\cdots x_0^{k_r-1}x_{z_r}). \qquad (3)
\end{aligned}$$

Therefore, for instance, the harmonic product

$$y_{1,z_1} * y_{1,z_2} = y_{1,z_1}y_{1,z_2} + y_{1,z_2}y_{1,z_1} + y_{2,z_1z_2}$$

corresponds to the identity

$$\mathrm{Li}_1(z_1)\,\mathrm{Li}_1(z_2) = \mathrm{Li}_{(1,1)}(z_1, z_2) + \mathrm{Li}_{(1,1)}(z_2, z_1) + \mathrm{Li}_2(z_1z_2).$$

The shuffle product

$$x_{z_1} ш\, x_{z_2} = x_{z_1}x_{z_2} + x_{z_2} ш x_{z_1}$$

corresponds to the identity

$$\mathrm{Li}_1(z_1)\,\mathrm{Li}_1(z_2) = \mathrm{Li}_{(1,1)}(z_1, z_1^{-1}z_2) + \mathrm{Li}_{(1,1)}(z_2, z_1z_2^{-1}).$$

Finite double shuffle relations is then expressed by the identity

$$\mathrm{Li}_{(1,1)}(z_1, z_2) + \mathrm{Li}_{(1,1)}(z_2, z_1) + \mathrm{Li}_2(z_1z_2) = \mathrm{Li}_{(1,1)}(z_1, z_1^{-1}z_2) + \mathrm{Li}_{(1,1)}(z_2, z_1z_2^{-1}).$$

Definition 1.5. Define the $\mathbb{Q}$-algebra homomorphism $\overline{\mathrm{reg}}_ш^T : \mathfrak{H}_{G,ш} \to \mathfrak{H}^0_{G,ш}[T]$ such that it is the identity on $\mathfrak{H}^0$ and maps x_0 to 0 and x_1 to T.
Define the $\mathbb{Q}$-algebra homomorphism $\mathrm{reg}_ш^T : \mathfrak{H}^1_{G,ш} \to \mathfrak{H}^0_{G,ш}[T]$ to be the restriction of $\overline{\mathrm{reg}}_ш^T$ to $\mathfrak{H}^1_G$ (see [Zha10, §3] and [IKZ06, §3] for $G = \{1\}$).

Lemma 1.6. *For $w = x_g w' \in \mathfrak{H}_G$, with $g \in \{0\} \sqcup G \smallsetminus \{1\}$, we have the following equality in $\mathfrak{H}^0_{G,ш}[T]$*

$$\overline{\mathrm{reg}}_ш^T\left(\frac{1}{1 - x_1u}w\right) = x_g\left(\frac{1}{1 + x_1u} ш\, \widetilde{\mathrm{reg}}_ш(w')\right)\exp(Tu),$$

where u is a formal parameter and $\widetilde{\mathrm{reg}}_ш : \mathfrak{H}_{G,ш} \to \mathfrak{H}^1_{G,ш}$ is the $\mathbb{Q}$-algebra homomorphism such that it is the identity on $\mathfrak{H}^1_G$ and maps x_0 to 0.

Proof. This is a generalization of [IKZ06, Proposition 8]. Its proof, and the one of [IKZ06, Proposition 7] which is used therein, can be adapted to our setup without any major change. □

Corollary 1.7. *For $w = x_g w' \in \mathfrak{H}^0_G$ we have*

$$\mathrm{reg}_ш^T\left(\frac{1}{1 - x_1u}w\right) = x_g\left(\frac{1}{1 + x_1u} ш w'\right)\exp(Tu).$$

Proof. This is an immediate consequence of Lemma 1.6. □

Lemma 1.8. *Let $Z_{\mathbf{A}} : \mathfrak{H}^0_{G,ш} \to \mathbf{A}$ be a $\mathbb{Q}$-algebra homomorphism. Then it extends to a unique $\mathbb{Q}$-algebra homomorphism*

$$\overline{Z}^ш_{\mathbf{A}} : \mathfrak{H}_{G,ш} \to \mathbf{A}[T]$$

which agrees with $Z_{\mathbf{A}}$ on $\mathfrak{H}^0_G$ and maps x_0 to 0 and x_1 to T.

Proof. It follows by considering the compositions

$$\mathfrak{H}_{G,ш} \xrightarrow{\overline{\mathrm{reg}}_ш^T} \mathfrak{H}^0_{G,ш}[T] \simeq \mathfrak{H}^0_{G,ш} \otimes \mathbb{Q}[T] \xrightarrow{Z_{\mathbf{A}}\otimes\mathrm{id}} \mathbf{A} \otimes \mathbb{Q}[T] \simeq \mathbf{A}[T]. \qquad (4)$$

□

Corollary 1.9. *Let $Z_{\mathbf{A}} : \mathfrak{H}^0_{G,ш} \to \mathbf{A}$ be a $\mathbb{Q}$-algebra homomorphism. Then it extends to a unique $\mathbb{Q}$-algebra homomorphism*

$$Z^{ш}_{\mathbf{A}} : \mathfrak{H}^1_{G,ш} \to \mathbf{A}[T]$$

which agrees with $Z_{\mathbf{A}}$ on $\mathfrak{H}^0_G$ and maps x_1 to T.

Proof. It follows by restriction of $\overline{Z}^{ш}_A$ from Lemma 1.8 to $\mathfrak{H}^1_G$. □

Corollary 1.10. *Let $Z_{\mathbf{A}} : \mathfrak{H}^0_{G,ш} \to \mathbf{A}$ be a $\mathbb{Q}$-algebra homomorphism. We have (equality of $\mathbb{Q}$-algebra morphisms $\mathfrak{H}_{G,ш} \to \mathbf{A}$)*

$$Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{ш} = \mathrm{ev}^{\mathbf{A}}_0 \circ \overline{Z}^{ш}_{\mathbf{A}}.$$

Proof. It follows from composition (4). □

We define the $\mathbf{A}$-module automorphism $\rho_{Z_{\mathbf{A}}}$ of $\mathbf{A}[T]$ such that

$$\rho_{Z_{\mathbf{A}}}(\exp(Tu)) = \Gamma_{Z_{\mathbf{A}}}(u)\exp(Tu),$$

where $\Gamma_{Z_{\mathbf{A}}}(u)$ is the power series defined by

$$\Gamma_{Z_{\mathbf{A}}}(u) := \exp\left(\sum_{n\geq 2} \frac{(-1)^n}{n} Z_{\mathbf{A}}(x_0^{n-1}x_1)u^n\right).$$

Definition 1.11. We define $\mathsf{EDS}(G)(\mathbf{A})$ to be the set of elements $Z_{\mathbf{A}} \in \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{H}^0_G, \mathbf{A})$ such that

(a) $Z_{\mathbf{A}} : \mathfrak{H}^0_{G,ш} \to \mathbf{A}$ is an algebra homomorphism;
(b) $\rho^{-1}_{Z_{\mathbf{A}}} \circ Z^{ш}_{\mathbf{A}} \circ \mathrm{q}_G^{-1} : \mathfrak{H}^1_{G,*} \to \mathbf{A}[T]$ is an algebra homomorphism.

An element of $\mathsf{EDS}(G)(A)$ is said to have the *extended double shuffle* property.

Remark 1.12. Assume that a $\mathbb{Q}$-linear map $Z_{\mathbf{A}} : \mathfrak{H}^0_G \to \mathbf{A}$ has the *finite double shuffle* property, that is, it is an algebra homomorphism for ш and $Z_{\mathbf{A}} \circ \mathrm{q}_G^{-1}$ is an algebra homomorphism for $*$. Similar as in Corollary 1.9 we can also extend the algebra homomorphism $Z_{\mathbf{A}} \circ \mathrm{q}_G^{-1} : \mathfrak{H}^0_{G,*} \to \mathbf{A}$ to an algebra homomorphism $Z^{*}_{\mathbf{A}} : \mathfrak{H}^1_{G,*} \to \mathbf{A}[T]$. If $Z_{\mathbf{A}}$ has the extended double shuffle property, then we have $Z^{*}_{\mathbf{A}} = \rho^{-1} \circ Z^{ш}_{\mathbf{A}}$.

Example 1.13. The $\mathbb{Q}$-linear map $Z_{\mathbb{C}} : \mathfrak{H}^0_{\mu_N} \to \mathbb{C}$ in Example 1.4 has the extended double shuffle property (cf. [Zha10, Theorem 2.9]) and therefore gives and element in $\mathsf{EDS}(\mu_N)(\mathbb{C})$.

1.2. **The scheme DMR of double shuffle and regularization relations.** Let $\mathbf{A}\langle\langle X_G\rangle\rangle$ be the free non-commutative associative series $\mathbf{A}$-algebra with unit over the alphabet X_G. It is the completion of the graded $\mathbf{A}$-algebra $\mathbf{A}\langle X_G\rangle \simeq \mathfrak{H}_G \otimes \mathbf{A}$ where elements of X_G has degree 1. One equips the algebra $\mathbf{A}\langle\langle X_G\rangle\rangle$ with the Hopf algebra coproduct $\widehat{\Delta}_{G,ш} : \mathbf{A}\langle\langle X_G\rangle\rangle \to \mathbf{A}\langle\langle X_G\rangle\rangle^{\hat{\otimes}2}$, called the *shuffle coproduct*, which is the unique $\mathbf{A}$-algebra homomorphism given by $\widehat{\Delta}_{G,ш}(x_g) = x_g \otimes 1 + 1 \otimes x_g$, for any $g \in G \sqcup \{0\}$ ([Rac02, §2.2.3]). Let then $\mathcal{G}(\mathbf{A}\langle\langle X_G\rangle\rangle)$ be the set of grouplike elements of $\mathbf{A}\langle\langle X_G\rangle\rangle$ for the coproduct $\widehat{\Delta}_{G,ш}$, i.e. the set

$$\mathcal{G}(\mathbf{A}\langle\langle X_G\rangle\rangle) = \{\Phi \in \mathbf{A}\langle\langle X_G\rangle\rangle^{\times} \mid \widehat{\Delta}_{G,ш}(\Phi) = \Phi \otimes \Phi\},$$

where $\mathbf{A}\langle\langle X_G\rangle\rangle^{\times}$ denotes the set of invertible elements of $\mathbf{A}\langle\langle X_G\rangle\rangle$.

Let $\mathbf{A}\langle\langle Y_G\rangle\rangle$ be the free non-commutative associative series $\mathbf{A}$-algebra with unit over the alphabet Y_G. It is the completion of the graded $\mathbf{A}$-algebra $\mathbf{A}\langle Y_G\rangle \simeq \mathfrak{H}^1_G \otimes \mathbf{A}$ where elements $y_{n,g}$ ($(n,g) \in \mathbb{Z}_{>0} \times G$) of Y_G has degree n. The algebra $\mathbf{A}\langle\langle Y_G\rangle\rangle$ can be viewed as a subalgebra of $\mathbf{A}\langle\langle X_G\rangle\rangle$ thanks to the mapping $y_{n,g} \mapsto x_0^{n-1}x_g$. One equips the algebra $\mathbf{A}\langle\langle Y_G\rangle\rangle$ with the

Hopf algebra coproduct $\widehat{\Delta}_{G,*} : \mathbf{A}\langle\langle Y_G\rangle\rangle \to \mathbf{A}\langle\langle Y_G\rangle\rangle^{\hat{\otimes}2}$, called the *harmonic coproduct*, which is the unique $\mathbf{A}$-algebra homomorphism such that ([Rac02, §2.3.1])

$$\widehat{\Delta}_{G,*}(y_{n,g}) = y_{n,g} \otimes 1 + 1 \otimes y_{n,g} + \sum_{\substack{k=1\\ h\in G}}^{n-1} y_{k,h} \otimes y_{n-k,gh^{-1}},$$

for any $(n, g) \in \mathbb{Z}_{>0} \times G$. Let then $\mathcal{G}(\mathbf{A}\langle\langle Y_G\rangle\rangle)$ be the set of grouplike elements of $\mathbf{A}\langle\langle Y_G\rangle\rangle$ for the coproduct $\widehat{\Delta}_{G,*}$, i.e. the set

$$\mathcal{G}(\mathbf{A}\langle\langle Y_G\rangle\rangle) = \{\Psi \in \mathbf{A}\langle\langle Y_G\rangle\rangle^\times \mid \widehat{\Delta}_{G,*}(\Psi) = \Psi \otimes \Psi\},$$

where $\mathbf{A}\langle\langle Y_G\rangle\rangle^\times$ denotes the set of invertible elements of $\mathbf{A}\langle\langle Y_G\rangle\rangle$.

Recall the direct sum decomposition (of $\mathbf{A}$-submodules) (see ([Rac02, §2.2.5])

$$\mathbf{A}\langle\langle X_G\rangle\rangle = \mathbf{A}\langle\langle Y_G\rangle\rangle \oplus \mathbf{A}\langle\langle X_G\rangle\rangle x_0.$$

Let then $\pi_{Y_G} : \mathbf{A}\langle\langle X_G\rangle\rangle = \mathbf{A}\langle\langle Y_G\rangle\rangle \oplus \mathbf{A}\langle\langle X_G\rangle\rangle x_0 \twoheadrightarrow \mathbf{A}\langle\langle Y_G\rangle\rangle$ be the projection from $\mathbf{A}\langle\langle X_G\rangle\rangle$ to $\mathbf{A}\langle\langle Y_G\rangle\rangle$, that is, the surjective $\mathbf{A}$-module homomorphism such that it is the identity on $\mathbf{A}\langle\langle Y_G\rangle\rangle$ and maps any element of $\mathbf{A}\langle\langle X_G\rangle\rangle x_0$ to 0.

Define the graded $\mathbf{A}$-module automorphism $\mathbf{q}_G$ of $\mathbf{A}\langle X_G\rangle$ as the composition

$$\mathbf{A}\langle X_G\rangle \simeq \mathbf{A} \otimes \mathfrak{H}_G \xrightarrow{\mathrm{id}_\mathbf{A} \otimes \mathrm{q}_G} \mathbf{A} \otimes \mathfrak{H}_G \to \mathbf{A}\langle X_G\rangle.$$

Its completion is an $\mathbf{A}$-module automorphism of $\mathbf{A}\langle\langle X_G\rangle\rangle$ that we shall denote $\mathbf{q}_G$ as well. Finally, define the map

$$\mathbf{A}\langle\langle X_G\rangle\rangle \to \mathbf{A}^{X_G^*},\ \Phi \mapsto \big((\Phi|w)\big)_{w\in X_G^*},$$

where $\Phi = \sum_{w\in X_G^*}(\Phi|w)w$.

Definition 1.14 ([Rac02, Definition 3.2.1])**.** We define[1] $\mathsf{DMR}(G)(\mathbf{A})$ to be the set[2] of $\Phi \in \mathbf{A}\langle\langle X_G\rangle\rangle$ with $(\Phi|x_0) = (\Phi|x_1) = 0$ such that

(a) $\widehat{\Delta}_{G,ш}(\Phi) = \Phi \otimes \Phi$ and $(\Phi|1) = 1$;
(b) $\widehat{\Delta}_{G,*}(\Phi_*) = \Phi_* \otimes \Phi_*$, where $\Phi_* := \Phi_{\mathrm{corr}}\ \pi_{Y_G}(\mathbf{q}_G(\Phi)) \in \mathbf{A}\langle\langle Y_G\rangle\rangle$ and

$$\Phi_{\mathrm{corr}} := \exp\left(\sum_{n\geq 2} \frac{(-1)^{n-1}}{n}(\Phi|x_0^{n-1}x_1)x_1^n\right).$$

Example 1.15. Set $\mathbf{A} = \mathbb{C}$, $G = \mu_N$ and $\zeta_N := e^{\frac{i2\pi}{N}}$. Define (see, for example, [Fur13])

$$\Phi^N_{\mathrm{KZ}} := 1 + \sum (-1)^r L_{(i_1,\ldots,i_r)}(\zeta_N^{l_2-l_1}, \ldots, \zeta_N^{l_r-l_{r-1}}, \zeta_N^{-l_r}) x_0^{i_r-1} x_{\zeta_N^{l_r}} \cdots x_0^{i_1-1} x_{\zeta_N^{l_1}} \\ + \text{(regularized terms)}$$

called the cyclotomic associator (see [Enr08]), which for $N = 1$ corresponds to the Drinfeld associator (see [Dri91]). It satisfies

$$\widehat{\Delta}_{G,ш}(\Phi^N_{\mathrm{KZ}}) = \Phi^N_{\mathrm{KZ}} \otimes \Phi^N_{\mathrm{KZ}} \qquad \text{and} \qquad \widehat{\Delta}_{G,*}(\Phi^N_{\mathrm{KZ}*}) = \Phi^N_{\mathrm{KZ}*} \otimes \Phi^N_{\mathrm{KZ}*},$$

which at the level of coefficients correspond to the double shuffle and regularization relations of cyclotomic multiple zeta values (see [Rac02]).

[1] The notation DMR is for "Double Mélange et Régularisation" which is French for "Double Shuffle and Regularization".
[2] In [Rac02], this is the set denoted by $\underline{\mathsf{DMR}}(G)(\mathbf{A})$.

1.3. **Comparison between $\mathsf{EDS}(G)(\mathbf{A})$ and $\mathsf{DMR}(G)(\mathbf{A})$.** Denote by $\mathrm{ev}_0^{\mathbf{A}} : \mathbf{A}[T] \to \mathbf{A}$ the evaluation map at $T=0$ and set

$$\overline{\mathrm{reg}}_{ш} := \mathrm{ev}_0^{\mathfrak{H}^0_{ш}} \circ \overline{\mathrm{reg}}_{ш}^T,$$

the unique $\mathbb{Q}$-algebra homomorphism $\mathfrak{H}_{ш} \to \mathfrak{H}^0_{ш}$ such that it is the identity on $\mathfrak{H}^0$ and maps both x_0 and x_1 to 0.
Recall from Definition A.2 the pairing

$$\begin{aligned}\mathbf{A}\langle\langle \mathcal{L}\rangle\rangle \otimes \mathbb{Q}\langle \mathcal{L}\rangle &\longrightarrow \mathbf{A}\\ \psi \otimes w &\longmapsto (\psi \mid w),\end{aligned}$$

where $\mathcal{L} = X_G$ or Y_G. Direct computations enables one to prove that

$$\mathbf{q}_G(\Phi) = \sum_{w\in\mathcal{L}^*} (\Phi \mid \mathrm{q}_G^{-1}(w))w. \tag{5}$$

Definition 1.16. Given a $\mathbb{Q}$-linear map $Z_{\mathbf{A}} : \mathfrak{H}^0_G \to \mathbf{A}$, we define the following element in $\mathbf{A}\langle\langle X_G\rangle\rangle$

$$\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}} := \sum_{w\in X_G^*} Z_{\mathbf{A}}(\overline{\mathrm{reg}}_{ш}(w))w\,.$$

Theorem 1.17. $Z_{\mathbf{A}} \in \mathsf{EDS}(G)(\mathbf{A}) \iff \Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}} \in \mathsf{DMR}(G)(\mathbf{A})$.

Proof. First, by definition of $\overline{\mathrm{reg}}_{ш}$ and by linearity of $Z_{\mathbf{A}}$, we have for $i\in\{0,1\}$,

$$(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}}|x_i) = Z_{\mathbf{A}}(\overline{\mathrm{reg}}_{ш}(x_i)) = 0.$$

Next, since $\overline{\mathrm{reg}}_{ш} : \mathfrak{H}_{G,ш} \to \mathfrak{H}^0_{G,ш}$ is an algebra homomorphism, the map $Z_{\mathbf{A}} : \mathfrak{H}^0_{G,ш} \to \mathbf{A}$ is an algebra homomorphism if and only if $Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш} : \mathfrak{H}_{G,ш} \to \mathbf{A}$ is an algebra homomorphism. Moreover, thanks to Proposition A.4, applied to $\mathcal{L} = X_G$, $\varphi = Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}$, $*_\diamond = ш$ and $\widehat{\Delta}_{*_\diamond} = \widehat{\Delta}_{G,ш}$, the map $Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш} : \mathfrak{H}_{G,ш} \to \mathbf{A}$ is an algebra homomorphism if and only if $\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}} \in \mathcal{G}(\mathbf{A}\langle\langle X_G\rangle\rangle)$. Finally, since $\mathrm{ev}_0^{\mathbf{A}}$ is an algebra homomorphism, the map $\rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш} \circ \mathrm{q}_G^{-1} : \mathfrak{H}^1_{G,*} \to \mathbf{A}[T]$ is an algebra homomorphism if and only if $\mathrm{ev}_0^{\mathbf{A}} \circ \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш} \circ \mathrm{q}_G^{-1} : \mathfrak{H}^1_{G,*} \to \mathbf{A}$ is an algebra homomorphism. Moreover, thanks to Proposition A.4, applied to $\mathcal{L} = Y_G$, $\varphi = \mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}$, $*_\diamond = *$ and $\widehat{\Delta}_{*_\diamond} = \widehat{\Delta}_{G,*}$, the map $\mathrm{ev}_0^{\mathbf{A}} \circ \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш} \circ \mathrm{q}_G^{-1} : \mathfrak{H}^1_{G,*} \to \mathbf{A}[T]$ is an algebra homomorphism if and only if $\Phi_{\mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}} \in \mathcal{G}(\mathbf{A}\langle\langle Y_G\rangle\rangle)$. Let us show that$\Phi_{\mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}} \in \mathcal{G}(\mathbf{A}\langle\langle Y_G\rangle\rangle)$ if and only if $(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_* \in \mathcal{G}(\mathbf{A}\langle\langle Y_G\rangle\rangle)$. In order to do so, it is enough to show that[3]

$$(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_* = \Phi_{\mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}}. \tag{6}$$

Step 1 Evaluation of $\Phi_{\mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}}$. We have

$$\Phi_{\mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}} = \sum_{w\in Y^*} \mathrm{ev}_0^{\mathbf{A}} \circ \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш} \circ \mathrm{q}_G^{-1}(w)\,w\,. \tag{7}$$

Step 2 Evaluation of $(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_* = (\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_{\mathrm{corr}}\ \pi_{Y_G}\left(\mathbf{q}_G(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})\right)$. In particular,

$$\begin{aligned}(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_{\mathrm{corr}} &= \exp\left(\sum_{n\geq 2}\frac{(-1)^{n-1}}{n}\left(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}}|x_0^{n-1}x_1\right)x_1^n\right)\\ &= \exp\left(-\sum_{n\geq 2}\frac{(-1)^{n}}{n}Z_{\mathbf{A}}\left(x_0^{n-1}x_1\right)x_1^n\right) = \Gamma_{Z_{\mathbf{A}}}^{-1}(x_1),\end{aligned}$$

[3]For the special case of $Z_{\mathbb{C}}$ in Example 1.4 this was shown in [YZ16, Theorem 3.13].

and

$$\pi_{Y_G}\circ\mathbf{q}_G(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})=\pi_{Y_G}\circ\mathbf{q}_G\left(\sum_{w\in X_G^*}Z_{\mathbf{A}}(\overline{\mathrm{reg}}_{ш}(w))w\right)=\sum_{w\in X_G^*}Z_{\mathbf{A}}(\overline{\mathrm{reg}}_{ш}(w))\ \pi_{Y_G}(\mathbf{q}_G(w))$$
$$=\sum_{w\in Y_G^*}Z_{\mathbf{A}}(\mathrm{reg}_{ш}(w))\,\mathbf{q}_G(w)=\sum_{w\in Y_G^*}Z_{\mathbf{A}}\circ\mathrm{reg}_{ш}\circ\mathrm{q}_G^{-1}(w)\,w,$$

where the second equality follows by linearity of $\pi_{Y_G}\circ\mathbf{q}_G$, the third one by definitions of π_{Y_G}, $\mathbf{q}_G$ and $\overline{\mathrm{reg}}_{ш}$ and the fourth one from identity (5). Therefore,

$$(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_*=\sum_{w\in Y_G^*}Z_{\mathbf{A}}\circ\mathrm{reg}_{ш}\circ\mathrm{q}_G^{-1}(w)\,\Gamma_{Z_{\mathbf{A}}}^{-1}(x_1)\,w. \tag{8}$$

In order to prove equality (6), it suffices to prove that

$$\big((\overline{\Phi}_{Z_{\mathbf{A}}})_*|x_1^m w_0\big)=\left(\Phi_{\mathrm{ev}_0\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}}|x_1^m w_0\right),$$

for any $m\in\mathbb{Z}_{\geq 0}$ and any $w_0\in\mathfrak{H}_G^0$.

Step 3 Evaluation of $\rho_{Z_{\mathbf{A}}}^{-1}$. Let u be a formal parameter.
Let us write

$$\Gamma_{Z_{\mathbf{A}}}^{-1}(u)=\sum_{l\geq 0}\gamma_l u^l, \tag{9}$$

where γ_a are elements in $\mathbf{A}$ expressed in terms of images by $Z_{\mathbf{A}}$. This implies that

$$\rho_{Z_{\mathbf{A}}}^{-1}\left(\frac{T^l}{l!}\right)=\sum_{j=0}^{l}\gamma_j\frac{T^{l-j}}{(l-j)!},$$

for any $l\in\mathbb{Z}_{\geq 0}$.

Step 4 Evaluation of $\mathrm{reg}_{ш}(x_1^n\,\mathrm{q}_G^{-1}(w_0))$ for any $n\in\mathbb{Z}_{\geq 0}$ and $w_0\in\mathfrak{H}_G^0$.
Recall that $\mathrm{q}_G^{-1}(w_0)\in\mathfrak{H}_G^0$, since $w_0\in\mathfrak{H}_G^0$. Therefore, one may write $\mathrm{q}_G^{-1}(w_0)=x_g w_0'$ for some $g\in(G\sqcup\{0\})\smallsetminus\{1\}$ and $w_0'\in\mathfrak{H}_G^1$. By Corollary 1.7 for the case $w=\mathrm{q}_G^{-1}(w_0)$ we obtain

$$\mathrm{reg}_{ш}^T\left(\sum_{n\geq 0}x_1^n u^n\,\mathrm{q}_G^{-1}(w_0)\right)=x_g\left(\sum_{k\geq 0}(-1)^k x_1^k u^k ш w_0'\right)\sum_{l\geq 0}\frac{T^l}{l!}u^l,$$

for a formal parameter u. This implies that

$$\mathrm{reg}_{ш}^T(x_1^n\,\mathrm{q}_G^{-1}(w_0))=\sum_{\substack{k,l\geq 0\\ k+l=n}}\frac{(-1)^k}{l!}x_g(x_1^k ш w_0')T^l \tag{10}$$

for any $n\in\mathbb{Z}_{\geq 0}$, and therefore

$$\mathrm{reg}_{ш}(x_1^n\,\mathrm{q}_G^{-1}(w_0))=(-1)^n x_g(x_1^n ш w_0'). \tag{11}$$

Step 5 Evaluation of $\big((\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_*|x_1^m w_0\big)$.
Injecting in (8) the expressions (9) and (11) from **Step 3** and **Step 4** respectively, we obtain

$$\big((\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_*|x_1^m w_0\big)=\sum_{\substack{k,l\geq 0\\ k+l=m}}Z_{\mathbf{A}}\circ\mathrm{reg}_{ш}\circ\mathrm{q}_G^{-1}(x_1^k w_0)\gamma_l=\sum_{\substack{k,l\geq 0\\ k+l=m}}Z_{\mathbf{A}}\circ\mathrm{reg}_{ш}(x_1^k\,\mathrm{q}_G^{-1}(w_0))\gamma_l$$
$$=\sum_{\substack{k,l\geq 0\\ k+l=m}}(-1)^k Z_{\mathbf{A}}\left(x_g(x_1^k ш w_0')\right)\gamma_l.$$

Step 6 Evaluation of $\left(\Phi_{\mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}}|x_1^m w_0\right)$. We have

$$\begin{aligned}\left(\Phi_{\mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}}|x_1^m w_0\right) &= \mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}(x_1^m w_0) = \mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}(x_1^m\,\mathrm{q}_G^{-1}(w_0))\\
&= \mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\left(\sum_{\substack{k,l\geq 0\\ k+l=m}}(-1)^k Z_{\mathbf{A}}\left(x_g(x_1^k ш w_0')\right)\frac{T^l}{l!}\right)\\
&= \mathrm{ev}_0^{\mathbf{A}}\left(\sum_{\substack{k,l\geq 0\\ k+l=m}}(-1)^k Z_{\mathbf{A}}\left(x_g(x_1^k ш w_0')\right)\sum_{j=0}^{l}\gamma_j\frac{T^{l-j}}{(l-j)!}\right)\\
&= \sum_{\substack{k,l\geq 0\\ k+l=m}}(-1)^k Z_{\mathbf{A}}\left(x_g(x_1^k ш w_0')\right)\gamma_l,\end{aligned}$$

where the first equality comes from expression (7) and the third one from the Composition (4) which interprets $Z_{\mathbf{A}}^{ш}$ as a composition involving $\mathrm{reg}_{ш}^T$ which enables us to use expression (10). Therefore, we have

$$\left((\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})_*|x_1^m w_0\right) = \sum_{\substack{k,l\geq 0\\ k+l=m}}(-1)^k Z_{\mathbf{A}}\left(x_g(x_1^k ш w_0')\right)\gamma_l = \left(\Phi_{\mathrm{ev}_0^{\mathbf{A}}\circ\rho_{Z_{\mathbf{A}}}^{-1}\circ Z_{\mathbf{A}}^{ш}\circ\mathrm{q}_G^{-1}}|x_1^m w_0\right),$$

which establishes equality (6). □

Theorem 1.18. *The map*

$$\begin{array}{ccc}\mathsf{EDS}(G)(\mathbf{A}) & \longrightarrow & \mathsf{DMR}(G)(\mathbf{A})\\ Z_{\mathbf{A}} & \longmapsto & \Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}}\end{array}\tag{12}$$

is bijective.

In order to prove this, we will need the following result:

Lemma 1.19. *Let $Z_{\mathbf{A}}:\mathfrak{H}_{G,ш}^0\to\mathbf{A}$ be a $\mathbb{Q}$-algebra homomorphism. Then $Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}:\mathfrak{H}_{G,ш}\to\mathbf{A}$ is the unique $\mathbb{Q}$-algebra homomorphism such that it agrees with $Z_{\mathbf{A}}$ on $\mathfrak{H}_G^0$ and maps x_0 to 0 and x_1 to 0.*

Proof. This follows by definition of $\overline{\mathrm{reg}}_{ш}$. □

Proof of Theorem 1.18. Let us start by showing that the map (12) is injective. Indeed, let $Z_{\mathbf{A}}$ and $Z'_{\mathbf{A}}\in\mathsf{EDS}(G)(\mathbf{A})$ such that $\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}}=\Phi_{Z'_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}}$. This implies that $Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}(w)=Z'_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}(w)$ for any $w\in X^*$. In particular, for any $w_0\in X^*\cap\mathfrak{H}_G^0$, one obtains $Z_{\mathbf{A}}(w)=Z'_{\mathbf{A}}(w)$. Then one concludes the equality by linearity.
Now, let us show surjectivity. Let $\Phi\in\mathsf{DMR}(G)(\mathbf{A})$. Define the linear map $Z_{\mathbf{A}}:\mathfrak{H}_G^0\to\mathbf{A}$ such that $w_0\mapsto(\Phi|w_0)$, for $w_0\in X^*\cap\mathfrak{H}_G^0$. Thanks to Theorem 1.17, the result follows if we show that $\Phi=\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}}$. This is equivalent to show that

$$(\Phi|w)=Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}(w),\tag{13}$$

for any $w\in X_G^*$. Let us consider the linear map $z_{\mathbf{A}}:\mathfrak{H}_G\to\mathbf{A}$ such that $w\mapsto(\Phi|w)$ for any $w\in X_G^*$. It is immediate that this map agrees with $Z_{\mathbf{A}}$ on $\mathfrak{H}_G^0$ and maps x_0 and x_1 to 0. In addition, since $\Phi\in\mathcal{G}(\mathbf{A}\langle\langle X_G\rangle\rangle)$, then $z_{\mathbf{A}}$ is an algebra homomorphism $\mathfrak{H}_{G,ш}\to\mathbf{A}$. On the other hand, the grouplikeness of Φ also implies that the map $Z_{\mathbf{A}}:\mathfrak{H}_{G,ш}^0\to\mathbf{A}$ is an algebra homomorphism. Therefore, thanks to Lemma 1.19, $Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}} : \mathfrak{H}_{G,\text{ш}} \to \mathbf{A}$ is the unique algebra morphism such that it is $Z_{\mathbf{A}}$ on $\mathfrak{H}_G^0$ and maps x_0 and x_1 to 0. This proves that $z_{\mathbf{A}} = Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}}$ and thus identity (13). □

2. Distribution relations

Throughout this section, G is a non-trivial group, $d \in \mathbb{Z}_{>0}$ is a divisor of the order of G and consider the subgroup

$$G^d := \{g^d \mid g \in G\}.$$

of G. This section extends algebraic relations between multiple polylogarithm values at roots of unity to include distribution relations. We begin by examining Racinet's approach, which encodes these additional relations into a subscheme $\mathsf{DMRD}(G)$ of $\mathsf{DMR}(G)$. Parallel to this, we introduce a subscheme $\mathsf{EDSD}(G)$ of $\mathsf{EDS}(G)$, which captures the distribution relations within the framework developed by Arakawa and Kaneko [AK04]. Thanks to the correspondence of schemes established in the previous section, we prove that these subschemes are indeed isomorphic. Finally, we apply this equivalent formalism to prove the conjecture of Zhao [Zha10, Conjeture 4.7] stated in the introduction.

2.1. **The homomorphisms i_d^*, p_*^d, $i_d^\sharp$ and $p_\sharp^d$.** Let $p^d : G \twoheadrightarrow G^d$ be the group homomorphism $g \mapsto g^d$ and $i_d : G^d \hookrightarrow G$ be the canonical inclusion.
Applying the functor given by Proposition-Definition B.1 (a) for $\phi = i_d$, we define the Hopf algebra homomorphism $i_d^* : (\mathbf{A}\langle\langle X_G\rangle\rangle, \widehat{\Delta}_{G,\text{ш}}) \to (\mathbf{A}\langle\langle X_{G^d}\rangle\rangle, \widehat{\Delta}_{G^d,\text{ш}})$. One checks that this homomorphism is explicitly given by

$$i_d^* : \begin{array}{l} x_0 \mapsto x_0 \\ x_g \mapsto \begin{cases} x_g & \text{if } g \in G^d \\ 0 & \text{otherwise} \end{cases} \end{array}$$

Applying the functor given by Proposition-Definition B.1 (b) for $\phi = p^d$, we define the Hopf algebra homomorphism $p_*^d : (\mathbf{A}\langle\langle X_G\rangle\rangle, \widehat{\Delta}_{G,\text{ш}}) \to (\mathbf{A}\langle\langle X_{G^d}\rangle\rangle, \widehat{\Delta}_{G^d,\text{ш}})$. One checks that this homomorphism is explicitly given by (see also [Rac02, §2.5.3] and [Zha10, §4])

$$p_*^d : \begin{array}{l} x_0 \mapsto d\, x_0 \\ x_g \mapsto x_{g^d} \end{array}$$

Applying the functor given by Proposition-Definition B.4 (a) for $\phi = i_d$, we define the algebra homomorphism $i_d^\sharp : \mathfrak{H}_{G^d} \to \mathfrak{H}_G$. One checks that this homomorphism is explicitly given by (see also [Rac02, §2.5.3] and [Zha10, §4])

$$i_d^\sharp : \begin{array}{l} x_0 \mapsto x_0 \\ x_h \mapsto x_h \end{array}$$

Applying the functor given by Proposition-Definition B.4 (b) for $\phi = p^d$, we define the algebra homomorphism $p_\sharp^d : \mathfrak{H}_{G^d} \to \mathfrak{H}_G$. One checks that this homomorphism is explicitly given by

$$p_\sharp^d : \begin{array}{l} x_0 \mapsto d\, x_0 \\ x_h \mapsto \displaystyle\sum_{g^d = h} x_g \end{array}$$

Recall from Definition A.2 the pairing $(-,-)_G : \mathbf{A}\langle\langle X_G\rangle\rangle \otimes \mathfrak{H}_G \to \mathbf{A}$ (resp. $(-,-)_{G^d} : \mathbf{A}\langle\langle X_{G^d}\rangle\rangle \otimes \mathfrak{H}_{G^d} \to \mathbf{A}$) for $\mathcal{L} = X_G$ (resp. $\mathcal{L} = X_{G^d}$).

Lemma 2.1. *We have*

(a) $(i_d^*(S_2), P_1)_{G^d} = (S_2, i_d^\sharp(P_1))_G$, *for* $S_2 \in \mathbf{A}\langle\langle X_G\rangle\rangle$ *and* $P_1 \in \mathfrak{H}_{G^d}$.

(b) $(p^d_*(S_1), P_2)_{G^d} = (S_1, p^d_\sharp(P_2))_G$*, for* $S_1 \in \mathbf{A}\langle\langle X_G\rangle\rangle$ *and* $P_2 \in \mathfrak{H}_{G^d}$*.*

Proof. This follows immediately by applying Lemma B.5 to $\phi = i_d$, $G_1 = G^d$ and $G_2 = G$ (resp. $\phi = p^d$, $G_1 = G$ and $G_2 = G^d$). □

2.2. **The schemes** DMRD **and** EDSD **of distribution relations.**

Definition 2.2 ([Rac02, Definition 3.2.1])**.** We define $\mathsf{DMRD}(G)(\mathbf{A})$ to be the set[4] of $\Phi \in \mathsf{DMR}(G)(\mathbf{A})$ such that for every divisor d of the order of G, we have

$$p^d_*(\Phi) = \exp\left(\sum_{g^d=1}(\Phi|x_g)x_1\right) i^*_d(\Phi).$$

Definition 2.3. A $\mathbb{Q}$-linear map $Z_\mathbf{A} : \mathfrak{H}^0_G \to \mathbf{A}$ satisfies *finite distribution* relations if for every divisor d of the order of G, we have (equality of $\mathbb{Q}$-linear maps $\mathfrak{H}^0_{G^d} \to \mathbf{A}$)

$$Z_\mathbf{A} \circ p^d_\sharp = Z_\mathbf{A} \circ i^\sharp_d.$$

Example 2.4. Let $N > 1$ be a positive integer. Finite distribution relations of multiple polylogarithm values at N^{th} roots of unity are expressed by (see [Gon01])

$$\mathrm{Li}_{(k_1,\dots,k_r)}(z_1,\dots,z_r) = d^{k_1+\cdots+k_r-r} \sum_{\substack{t_i^d=z_i\\ 1\le i\le r}} \mathrm{Li}_{(k_1,\dots,k_r)}(t_1,\dots,t_r), \tag{14}$$

for any divisor d of N. Recall from Example 1.4 the $\mathbb{Q}$-linear map $Z_\mathbb{C} : \mathfrak{H}^0_{\mu_N} \to \mathbb{C}$. For any divisor d of N and any word $x_0^{k_1-1}x_{z_1}\cdots x_0^{k_r-1}x_{z_r}$ of $\mathfrak{H}^0_{\mu_N^d}$, we have

$$Z_\mathbb{C} \circ \mathrm{q}^{-1}_{\mu_N} \circ p^d_\sharp(x_0^{k_1-1}x_{z_1}\cdots x_0^{k_r-1}x_{z_r}) = d^{k_1+\cdots+k_r-r} \sum_{\substack{t_i^d=z_i\\ 1\le i\le r}} Z_\mathbb{C} \circ \mathrm{q}^{-1}_{\mu_N}(x_0^{k_1-1}x_{t_1}\cdots x_0^{k_r-1}x_{t_r}),$$

which is equal to the right hand side of (14) thanks to (3). On the other hand,

$$Z_\mathbb{C} \circ \mathrm{q}^{-1}_{\mu_N} \circ i^\sharp_d(x_0^{k_1-1}x_{z_1}\cdots x_0^{k_r-1}x_{z_r}) = Z_\mathbb{C} \circ \mathrm{q}^{-1}_{\mu_N}(x_0^{k_1-1}x_{z_1}\cdots x_0^{k_r-1}x_{z_r}),$$

which is the left hand side of (14) thanks to (3). Therefore, equality (14) is satisfied if and only if

$$Z_\mathbb{C} \circ \mathrm{q}^{-1}_{\mu_N} \circ p^d_\sharp = Z_\mathbb{C} \circ \mathrm{q}^{-1}_{\mu_N} \circ i^\sharp_d.$$

Thanks to Corollary B.7 (30) (resp. (31)) applied to $\phi = i_d$, $G_1 = \mu_N$ and $G_2 = \mu_N^d$ (resp. $\phi = p^d$, $G_1 = \mu_N^d$ and $G_2 = \mu_N$), this is equivalent to

$$Z_\mathbb{C} \circ p^d_\sharp \circ \mathrm{q}^{-1}_{\mu_N^d} = Z_\mathbb{C} \circ i^\sharp_d \circ \mathrm{q}^{-1}_{\mu_N^d}.$$

Hence, equality (14) is satisfied if and only if

$$Z_\mathbb{C} \circ p^d_\sharp = Z_\mathbb{C} \circ i^\sharp_d.$$

Definition 2.5. For a $\mathbb{Q}$-linear map $Z_\mathbf{A} : \mathfrak{H}^0_G \to \mathbf{A}$, we define the $\mathbf{A}$-module automorphism $\sigma_{Z_\mathbf{A}}$ of $\mathbf{A}[T]$ such that

$$\sigma_{Z_\mathbf{A}}(\exp(Tu)) = \exp\left(\sum_{\substack{g^d=1\\ g\neq 1}} Z_\mathbf{A}(x_g)u\right)\exp(Tu),$$

for a formal parameter u.

[4] In [Rac02], this is the set denoted by $\underline{\mathsf{DMRD}}(G)(\mathbf{A})$.

Let us write

$$\exp\left(\sum_{\substack{g^d=1\\g\neq 1}} Z_{\mathbf{A}}(x_g)u\right) = \sum_{l\geq 0}\delta_l u^l, \tag{15}$$

where $\delta_0 = 1$, $\delta_1 = \sum_{\substack{g^d=1\\g\neq 1}} Z_{\mathbf{A}}(x_g)$ and $\delta_l = \frac{\delta_1^l}{l!}$ for $l \geq 0$. It follows that

$$\sigma_{Z_{\mathbf{A}}}\left(\frac{T^l}{l!}\right) = \sum_{j=0}^{l}\delta_j\frac{T^{l-j}}{(l-j)!}. \tag{16}$$

Additionally, for a word $w = x_h w'$ with $h \in (G^d \sqcup \{0\}) \setminus \{1\}$ and $w' \in X_{G^d}^*$, we obtain from Lemma 1.6 that

$$\overline{\mathrm{reg}}_{ш}^T\left(\sum_{n\geq 0} x_1^n u^n w\right) = x_h\left(\sum_{k\geq 0}(-1)^k x_1^k u^k \, ш \, \widetilde{\mathrm{reg}}_{ш}(w')\right)\sum_{l\geq 0}\frac{T^l}{l!}u^l.$$

In particular, this implies that

$$\overline{\mathrm{reg}}_{ш}^T(x_1^m w) = \sum_{\substack{k,l\geq 0\\k+l=m}}\frac{(-1)^k}{l!}x_h(x_1^k \, ш \, \widetilde{\mathrm{reg}}_{ш}(w'))T^l, \tag{17}$$

and therefore

$$\overline{\mathrm{reg}}_{ш}(x_1^m w) = (-1)^m x_h(x_1^m \, ш \, \widetilde{\mathrm{reg}}_{ш}(w')). \tag{18}$$

Definition 2.6. A $\mathbb{Q}$-algebra homomorphism $Z_{\mathbf{A}} : \mathfrak{H}_{G,ш}^0 \to \mathbf{A}$ satisfies *regularized distribution* relations if for every divisor d of the order of G, we have the following equality of $\mathbb{Q}$-linear maps $\mathfrak{H}_{G^d} \to \mathbf{A}[T]$

$$\overline{Z}_{\mathbf{A}}^{ш} \circ p_\sharp^d = \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp.$$

Proposition 2.7. *An algebra homomorphism* $Z_{\mathbf{A}} : \mathfrak{H}_{ш}^0 \to \mathbf{A}$ *satisfies regularized distribution relations if and only if for every divisor* d *of the order of* G*, we have an equality of* $\mathbb{Q}$*-linear maps* $\mathfrak{H}_{G^d} \to \mathbf{A}$

$$\mathrm{ev}_0^{\mathbf{A}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ p_\sharp^d = \mathrm{ev}_0^{\mathbf{A}} \circ \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp.$$

Proof. ($\Rightarrow$) is immediate. Let us prove ($\Leftarrow$). In the following, we fix d a divisor of the order of G.

Step 1 For any $m \in \mathbb{Z}_{\geq 0}$, let us show that

$$\overline{Z}_{\mathbf{A}}^{ш} \circ p_\sharp^d(x_1^m) = \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp(x_1^m).$$

We have

$$\begin{aligned}\sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp(x_1^m) &= \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp\left(\frac{x_1^{ш m}}{m!}\right) = \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш}\left(\frac{x_1^{ш m}}{m!}\right)\\ &= \sigma_{Z_{\mathbf{A}}}\left(\frac{\overline{Z}_{\mathbf{A}}^{ш}(x_1)^m}{m!}\right) = \sigma_{Z_{\mathbf{A}}}\left(\frac{T^m}{m!}\right) = \sum_{j=0}^{m}\delta_j\frac{T^{m-j}}{(m-j)!},\end{aligned}$$

where the first equality comes from $x_1^{ш m} = x_1 \, ш x_1 \, ш \cdots ш \, x_1 = m!\, x_1^m$, the third one from the fact that $\overline{Z}_{\mathbf{A}}^{ш} : \mathfrak{H}_{G,ш} \to \mathbf{A}[T]$ is an algebra homomorphism and the last one from equality (16).

On the other hand, we have

$$\begin{aligned}\overline{Z}^{ш}_{\mathbf{A}}\circ p^d_\sharp(x_1^m) &= \overline{Z}^{ш}_{\mathbf{A}}\circ p^d_\sharp\left(\frac{x_1^{шm}}{m!}\right) = \overline{Z}^{ш}_{\mathbf{A}}\left(\frac{p^d_\sharp(x_1)^{шm}}{m!}\right) = \frac{1}{m!}\left(\overline{Z}^{ш}_{\mathbf{A}}\circ p^d_\sharp(x_1)\right)^m\\ &= \frac{1}{m!}\left(\overline{Z}^{ш}_{\mathbf{A}}\left(\sum_{g^d=1} x_g\right)\right)^m = \frac{1}{m!}\left(\overline{Z}^{ш}_{\mathbf{A}}\left(x_1+\sum_{\substack{g^d=1\\ g\neq 1}} x_g\right)\right)^m\\ &= \frac{1}{m!}\left(T+Z_{\mathbf{A}}\left(\sum_{\substack{g^d=1\\ g\neq 1}} x_g\right)\right)^m = \frac{1}{m!}(\delta_0 T+\delta_1)^m\\ &= \frac{1}{m!}\sum_{j=0}^m \frac{m!}{j!(m-j)!}\delta_1^j T^{m-j} = \sum_{j=0}^m \delta_j \frac{T^{m-j}}{(m-j)!},\end{aligned}$$

where the second equality comes from the fact that $p^d_\sharp : \mathfrak{H}_{G^d,ш}\to\mathfrak{H}_{G,ш}$ is an algebra homomorphism thanks to Corollary B.6 (a) and the last equality comes from the identity $\delta_j = \frac{\delta_1^j}{j!}$ for $j\geq 1$.

Step 2 For a word w of $\mathfrak{H}_{G^d,ш}$ that does not start with x_1, let us show that

$$Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}\circ p^d_\sharp(w) = Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}\circ i^\sharp_d(w).$$

By assumption on w, let us write $w=x_h w'$ with $h\in(G^d\cup\{0\})\smallsetminus\{1\}$ and $w'\in X^*_{G^d}$. We have

$$\begin{aligned}Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}\circ i^\sharp_d(w) &= Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}\circ i^\sharp_d(x_h w') = Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}(x_h w')\\ &= Z_{\mathbf{A}}\left(x_h(1\,ш\,\widetilde{\mathrm{reg}}_{ш}(w'))\right) = Z_{\mathbf{A}}\left(x_h\widetilde{\mathrm{reg}}_{ш}(w')\right),\end{aligned}$$

where the third equality comes from (18). On the other hand,

$$\begin{aligned}Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}\circ p^d_\sharp(w) &= \mathrm{ev}_0^{\mathbf{A}}\circ\overline{Z}^{ш}_{\mathbf{A}}\circ p^d_\sharp(w) = \mathrm{ev}_0^{\mathbf{A}}\circ\sigma_{Z_{\mathbf{A}}}\circ\overline{Z}^{ш}_{\mathbf{A}}\circ i^\sharp_d(w)\\ &= \mathrm{ev}_0^{\mathbf{A}}\circ\sigma_{Z_{\mathbf{A}}}\circ\overline{Z}^{ш}_{\mathbf{A}}\circ i^\sharp_d(x_h w') = \mathrm{ev}_0^{\mathbf{A}}\circ\sigma_{Z_{\mathbf{A}}}\circ\overline{Z}^{ш}_{\mathbf{A}}(x_h w')\\ &= \mathrm{ev}_0^{\mathbf{A}}\circ\sigma_{Z_{\mathbf{A}}}\left(Z_{\mathbf{A}}(x_h(1\,ш\,\widetilde{\mathrm{reg}}_{ш}(w')))\right) = \mathrm{ev}_0^{\mathbf{A}}\left(Z_{\mathbf{A}}(x_h\,\widetilde{\mathrm{reg}}_{ш}(w'))\right)\\ &= Z_{\mathbf{A}}\left(x_h\widetilde{\mathrm{reg}}_{ш}(w')\right),\end{aligned}$$

where the first equality comes from Corollary 1.10, the second one from assumption on $Z_{\mathbf{A}}$ and the fifth one from expression (18).

Step 3 For a word v of $\mathfrak{H}_{G,ш}$ that does not start with x_1, let us show that

$$\overline{Z}^{ш}_{\mathbf{A}}(v) = Z_A\circ\overline{\mathrm{reg}}_{ш}(v).$$

By assumption on v, let us write $v=x_g v'$ with $g\in(G\cup\{0\})\smallsetminus\{1\}$ and $v'\in X^*_G$. We have

$$\overline{Z}^{ш}_{\mathbf{A}}(v) = \overline{Z}^{ш}_{\mathbf{A}}(x_h v') = Z_{\mathbf{A}}(x_h\widetilde{\mathrm{reg}}_{ш}(v')) = Z_A\circ\overline{\mathrm{reg}}_{ш}(x_h v') = Z_A\circ\overline{\mathrm{reg}}_{ш}(v),$$

where the second equality comes from the composition (4) which interprets $\overline{Z}^{ш}_{\mathbf{A}}$ as a composition involving $\overline{\mathrm{reg}}^T_{ш}$ and enables the use of expression (17) and the third equality comes from expression (18).

Step 4 For a word w of $\mathfrak{H}_{G^d,ш}$ that does not start with x_1, let us show that

$$\overline{Z}^{ш}_{\mathbf{A}}\circ p^d_\sharp(w) = \overline{Z}^{ш}_{\mathbf{A}}\circ i^\sharp_d(w).$$

Since $w\in X^*_{G^d}$ does not start with x_1, so is $i^\sharp_d(w)$. Therefore, from **Step 3**, it follows that

$$\overline{Z}^{ш}_{\mathbf{A}}\circ i^\sharp_d(w) = Z_A\circ\overline{\mathrm{reg}}_{ш}\circ i^\sharp_d(w). \tag{19}$$

On the other hand, since $w \in X_{G^d}^*$ does not start with x_1, $p_\sharp^d(w)$ is a linear combination of words in X_G^* that do not start with x_1. Therefore, from **Step 3**, it follows that

$$\overline{Z}_{\mathbf{A}}^{\text{ш}} \circ p_\sharp^d(w) = Z_A \circ \overline{\mathrm{reg}}_{\text{ш}} \circ p_\sharp^d(w). \tag{20}$$

Then

$$\overline{Z}_{\mathbf{A}}^{\text{ш}} \circ p_\sharp^d(w) = Z_A \circ \overline{\mathrm{reg}}_{\text{ш}} \circ p_\sharp^d(w) = Z_A \circ \overline{\mathrm{reg}}_{\text{ш}} \circ i_d^\sharp(w) = \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ i_d^\sharp(w),$$

where the first equality follows from (20); the second one from **Step 2** and the last one from (19).

Step 5 For $m \in \mathbb{Z}_{\geq 0}$ and a word w of $\mathfrak{H}_{G^d,\text{ш}}$ that does not start with x_1, let us show that

$$\overline{Z}_{\mathbf{A}}^{\text{ш}} \circ p_\sharp^d(x_1^m \text{ ш } w) = \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ i_d^\sharp(x_1^m \text{ ш } w).$$

We have

$$\begin{aligned}
\overline{Z}_{\mathbf{A}}^{\text{ш}} \circ p_\sharp^d(x_1^m \text{ ш } w) &= \overline{Z}_{\mathbf{A}}^{\text{ш}} \left(p_\sharp^d(x_1^m) \text{ ш } p_\sharp^d(w)\right) = \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ p_\sharp^d(x_1^m)\ \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ p_\sharp^d(w) \\
&= \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ i_d^\sharp(x_1^m)\ \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ i_d^\sharp(w) = \sigma_{Z_{\mathbf{A}}} \left(\overline{Z}_{\mathbf{A}}^{\text{ш}} \circ i_d^\sharp(x_1^m)\ \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ i_d^\sharp(w)\right) \\
&= \sigma_{Z_{\mathbf{A}}} \left(\overline{Z}_{\mathbf{A}}^{\text{ш}} \left(i_d^\sharp(x_1^m) \text{ ш } i_d^\sharp(w)\right)\right) = \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ i_d^\sharp(x_1^m \text{ ш } w),
\end{aligned}$$

where the first (resp. last) equality comes from the fact that $p_\sharp^d : \mathfrak{H}_{G^d,\text{ш}} \to \mathfrak{H}_{G,\text{ш}}$ (resp. $i_d^\sharp : \mathfrak{H}_{G^d,\text{ш}} \to \mathfrak{H}_{G,\text{ш}}$) is an algebra homomorphism thanks to Corollary B.6 (a), the second and fifth ones from the fact that $\overline{Z}_{\mathbf{A}}^{\text{ш}} : \mathfrak{H}_{\text{ш}} \to \mathbf{A}[T]$ is an algebra homomorphism and the third one from **Step 1** and **Step 4**, the fourth one from the $\mathbf{A}$-linearity of $\sigma_{Z_{\mathbf{A}}}$ and the fact that $\overline{Z}_{\mathbf{A}}^{\text{ш}} \circ i_d^\sharp(w) \in \mathbf{A}$ thanks to **Step 3**.

Finally, since all words in $\mathfrak{H}_{G^d}$ can be generated by the elements $x_1^m \text{ ш } w$, with $m \in \mathbb{Z}_{\geq 0}$ and $w \in X_{G^d}^*$ that does not start with x_1, the equality of **Step 5** suffices to prove that

$$Z_{\mathbf{A}} \circ p_\sharp^d = Z_{\mathbf{A}} \circ i_d^\sharp,$$

which is the wanted result. □

Definition 2.8. We define $\mathsf{EDSD}(G)(\mathbf{A})$ to be the set of elements $Z_{\mathbf{A}} \in \mathsf{EDS}(G)(\mathbf{A})$ such that the algebra homomorphism $Z_{\mathbf{A}} : \mathfrak{H}_{G,\text{ш}}^0 \to \mathbf{A}$ satisfies regularized distribution relations.

Theorem 2.9. *The bijection* $\mathsf{EDS}(G)(\mathbf{A}) \to \mathsf{DMR}(G)(\mathbf{A})$, $Z_{\mathbf{A}} \mapsto \Phi_{Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}}}$ *restricts to a bijection*

$$\mathsf{EDSD}(G)(\mathbf{A}) \longrightarrow \mathsf{DMRD}(G)(\mathbf{A}).$$

Proof. Thanks to Theorem 1.18, this is equivalent to show that for $Z_{\mathbf{A}} \in \mathsf{EDS}(G)(\mathbf{A})$

$$Z_{\mathbf{A}} \in \mathsf{EDSD}(G)(\mathbf{A}) \Longleftrightarrow \Phi_{Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}}} \in \mathsf{DMRD}(G)(\mathbf{A}).$$

In the following, we fix a divisor d of the order of G. We have

$$\begin{aligned}
p_*^d(\Phi_{Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}}}) &= \sum_{w \in X_G^*} Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}}(w) p_*^d(w) = \sum_{w \in X_{G^d}^*} Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}} \circ p_\sharp^d(w)\, w \\
&= \sum_{w \in X_{G^d}^*} \mathrm{ev}_0^{\mathbf{A}} \circ \overline{Z}_{\mathbf{A}}^{\text{ш}} \circ p_\sharp^d(w)\, w,
\end{aligned}$$

where the second equality follows from Lemma B.5 applied to $\phi = p^d$, and the third one from Corollary 1.10. On the other hand, we have

$$\exp\left(\sum_{g^d=1} (\Phi_{Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}}} | x_g) x_1\right) i_d^*(\Phi_{Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}}}) = \exp\left(\sum_{\substack{g^d=1 \\ g \neq 1}} Z_{\mathbf{A}}(x_g) x_1\right) \sum_{w \in X_G^*} Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{\text{ш}}(w) i_d^*(w)$$

$$= \sum_{w\in X_{G^d}^*} Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_{Ш} \circ i_d^\sharp(w) \exp\left(\sum_{\substack{g^d=1\\ g\neq 1}} Z_{\mathbf{A}}(x_g)x_1\right) w,$$

where the second equality comes from Lemma B.5 applied to $\phi = i_d$.
Let $m \geq 0$ and $w \in X_{G^d}^*$ that does not starting with x_1. We have

$$\left(p_*^d(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_Ш})\middle| x_1^m w\right) = \mathrm{ev}_0^{\mathbf{A}} \circ \overline{Z}_{\mathbf{A}}^{Ш} \circ p_\sharp^d(x_1^m w). \tag{21}$$

On the other hand, we have

$$\left(\exp\left(\sum_{g^d=1}(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_Ш}|x_g)x_1\right) i_d^*(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_Ш})\middle| x_1^m w\right) = \sum_{\substack{k+l=m\\ k,l\geq 0}} Z_{\mathbf{A}} \circ \overline{\mathrm{reg}}_Ш \circ i_d^\sharp(x_1^k w)\delta_l$$

$$= \sum_{\substack{k+l=m\\ k,l\geq 0}} Z_{\mathbf{A}}\left((-1)^k x_h\left(x_1^k Ш \widetilde{\mathrm{reg}}_Ш(w')\right)\right)\delta_l, \tag{22}$$

where the first equality comes from identity (22) and expression (15) and the second one from expression (18).

Case 1 Assume that $Z_{\mathbf{A}} \in \mathsf{EDSD}(G)(\mathbf{A})$. Then, for any divisor d of the order of G, $m \geq 0$ and $w \in X_{G^d}^*$ that does not starting with x_1, we have

$$\begin{aligned}\left(p_*^d(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_Ш})\middle| x_1^m w\right) &= \mathrm{ev}_0^{\mathbf{A}} \circ \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{Ш} \circ i_d^\sharp(x_1^m w) = \mathrm{ev}_0^{\mathbf{A}} \circ \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{Ш}(x_1^m w)\\
&= \mathrm{ev}_0^{\mathbf{A}} \circ \sigma_{Z_{\mathbf{A}}}\left(\sum_{\substack{k+l=m\\ k,l\geq 0}} \frac{(-1)^k}{l!} Z_{\mathbf{A}}\left(x_h\left(x_1^k Ш \widetilde{\mathrm{reg}}_Ш(w')\right)\right) T^l\right)\\
&= \mathrm{ev}_0^{\mathbf{A}}\left(\sum_{\substack{k+l=m\\ k,l\geq 0}} (-1)^k Z_{\mathbf{A}}\left(x_h\left(x_1^k Ш \widetilde{\mathrm{reg}}_Ш(w')\right)\right)\sigma_{Z_{\mathbf{A}}}\left(\frac{T^l}{l!}\right)\right)\\
&= \mathrm{ev}_0^{\mathbf{A}}\left(\sum_{\substack{k+l=m\\ k,l\geq 0}} (-1)^k Z_{\mathbf{A}}\left(x_h\left(x_1^k Ш \widetilde{\mathrm{reg}}_Ш(w')\right)\right)\sum_{j=0}^{l}\delta_j \frac{T^{l-j}}{(l-j)!}\right)\\
&= \sum_{\substack{k+l=m\\ k,l\geq 0}} (-1)^k Z_{\mathbf{A}}\left(x_h\left(x_1^k Ш \widetilde{\mathrm{reg}}_Ш(w')\right)\right)\delta_l,\end{aligned}$$

where the first equality comes from (21) and assumption of $Z_{\mathbf{A}}$, third one from from the composition (4) which interprets $\overline{Z}_{\mathbf{A}}^{Ш}$ as a composition involving $\overline{\mathrm{reg}}_Ш^T$, which enables us to use expression (17) and the fifth equality comes from (16). This implies that

$$\left(p_*^d(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_Ш})\middle| x_1^m w\right) = \left(\exp\left(\sum_{g^d=1}(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_Ш}|x_g)x_1\right) i_d^*(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_Ш})\middle| x_1^m w\right),$$

which proves that $\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_Ш} \in \mathsf{DMRD}(G)(\mathbf{A})$.

Case 2 Assume that $Z_{\mathbf{A}} \notin \mathsf{EDSD}(G)(A)$. Therefore, there exists divisor d of the order of G, $m \geq 0$ and $w \in \mathfrak{H}_{G^d}$ that does not start with x_1 such that

$$\overline{Z}_{\mathbf{A}}^{Ш} \circ p_\sharp^d(x_1^m w) \neq \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{Ш} \circ i_d^\sharp(x_1^m w).$$

From Proposition 2.7, it follows that

$$\mathrm{ev}_0^{\mathbf{A}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ p_\sharp^d(x_1^m w) \neq \mathrm{ev}_0^{\mathbf{A}} \circ \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp(x_1^m w).$$

On the other hand,

$$\left(\exp\left(\sum_{g^d=1}(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}}|x_g)x_1\right) i_d^*(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})\middle| x_1^m w\right) = \sum_{\substack{k+l=m\\ k,l\geq 0}} \mathrm{ev}_0^{\mathbf{A}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp(x_1^k w)\,\delta_l$$

$$= \sum_{\substack{k+l=m\\ k,l\geq 0}} \mathrm{ev}_0^{\mathbf{A}} \left(\sum_{\substack{k_1+k_2=k\\ k_1,k_2\geq 0}} \frac{(-1)^{k_1}}{k_2} Z_{\mathbf{A}}(x_h(x_1^{k_1} ш \widetilde{\mathrm{reg}}_{ш}(w')))T^{k_2} \right) \delta_l$$

$$= \sum_{\substack{k+l=m\\ k,l\geq 0}} (-1)^k Z_{\mathbf{A}}(x_h(x_1^k ш \widetilde{\mathrm{reg}}_{ш}(w')))\,\delta_l = \mathrm{ev}_0^{\mathbf{A}} \circ \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp(x_1^m w).$$

Therefore, thanks to this equality, identity (21) and the assumption on $Z_{\mathbf{A}}$, it follows that

$$\left(p_*^d(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})\middle| x_1^m w\right) \neq \left(\exp\left(\sum_{g^d=1}(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}}|x_g)x_1\right) i_d^*(\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}})\middle| x_1^m w\right),$$

which implies that $\Phi_{Z_{\mathbf{A}}\circ\overline{\mathrm{reg}}_{ш}} \notin \mathsf{DMRD}(G)(\mathbf{A})$.

□

2.3. **Proof of Theorem I.** We now prove Theorem I stated in the introduction. For this, we establish the equivalence of Theorem I and Theorem 2.10 in Proposition 2.11. After that we prove Theorem 2.10. These results collectively yield a proof of Theorem I.

Theorem 2.10. *Let $Z_{\mathbf{A}} : \mathfrak{H}_G^0 \to \mathbf{A}$ be a $\mathbb{Q}$-linear map and d a divisor of the order of G such that:*

(i) $Z_{\mathbf{A}} \in \mathsf{EDS}(G)(\mathbf{A})$*;*
(ii) $Z_{\mathbf{A}} \circ p_\sharp^d(x_h) = Z_{\mathbf{A}} \circ i_d^\sharp(x_h)$*, for any* $h \in G^d \smallsetminus \{1\}$*;*
(iii) $Z_{\mathbf{A}} \circ p_\sharp^d(x_{h_1}x_{h_2}) = Z_{\mathbf{A}} \circ i_d^\sharp(x_{h_1}x_{h_2})$*, for any* $h_1 \in G^d \smallsetminus \{1\}$ *and* $h_2 \in G^d$*;*

where (ii) and (iii) are equalities in $\mathbf{A}$*. Then, the following equality in* $\mathbf{A}[T]$ *holds*

$$\overline{Z}_{\mathbf{A}}^{ш} \circ p_\sharp^d(x_{h_1}x_{h_2}) = \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}_{\mathbf{A}}^{ш} \circ i_d^\sharp(x_{h_1}x_{h_2}), \tag{23}$$

for any $h_1, h_2 \in G^d \sqcup \{0\}$*.*

Proposition 2.11. *Theorem I is equivalent to Theorem 2.10.*

Proof. From Theorem 2.9, it follows that:

(a) assumption (i) is equivalent to the extended double shuffle relations;
(b) assumption (ii) is equivalent to weight one finite distribution relations;
(c) assumption (iii) is equivalent to depth two finite distribution relations;
(d) identity (23) is equivalent to regularized distribution relations.

□

We then dedicate the rest of this section entirely to the proof of Theorem 2.10. First, let us introduce the following elements

Definition 2.12. Let d be a divisor of the order of G. For $h, h_2 \in G^d$, $h_1 \in G^d \smallsetminus \{1\}$, $g, g_1, g_2 \in G \smallsetminus \{1\}$ define[5] the following elements of $\mathfrak{H}_G^0$:

$$\begin{aligned}
\mathrm{FDT}_{d,1}(h) &:= d \sum_{g^d=h} x_0 x_g - x_0 x_h, \\
\mathrm{FDT}_{d,2}(h_1, h_2) &:= \sum_{g_1^d=h_1} \sum_{g_2^d=h_2} x_{g_1} x_{g_2} - x_{h_1} x_{h_2}, \\
\mathrm{FDS}(g_1, g_2) &:= x_0 x_{g_1 g_2} + x_{g_1} x_{g_1 g_2} + x_{g_2} x_{g_1 g_2} - x_{g_1} x_{g_2} - x_{g_2} x_{g_1}, \\
\mathrm{RDS}(g) &:= x_0 x_g + x_g x_g - x_g x_1.
\end{aligned}$$

Lemma 2.13. *Let $Z_{\mathbf{A}} : \mathfrak{H}_G^0 \to \mathbf{A}$ be a $\mathbb{Q}$-linear map and d be a divisor of the order of G. For $h, h_2 \in G^d$, $h_1 \in G^d \smallsetminus \{1\}$, $g, g_1, g_2 \in G \smallsetminus \{1\}$, we have*

(a) $\mathrm{FDT}_{d,1}(h) \in \ker(Z_{\mathbf{A}}) \iff Z_{\mathbf{A}} \circ p_\sharp^d(x_0 x_h) = Z_{\mathbf{A}} \circ i_d^\sharp(x_0 x_h)$;

(b) $\mathrm{FDT}_{d,2}(h_1, h_2) \in \ker(Z_{\mathbf{A}}) \iff Z_{\mathbf{A}} \circ p_\sharp^d(x_{h_1} x_{h_2}) = Z_{\mathbf{A}} \circ i_d^\sharp(x_{h_1} x_{h_2})$;

(c) $\mathrm{FDS}(g_1, g_2) \in \ker(Z_{\mathbf{A}}) \iff Z_{\mathbf{A}} \circ \mathrm{q}_G^{-1}(x_{g_1} * x_{g_2}) = Z_{\mathbf{A}}(x_{g_1} ш x_{g_2})$;

(d) Assume $Z_{\mathbf{A}} : \mathfrak{H}_{G,ш}^0 \to \mathbf{A}$ is an algebra homomorphism. Then

$$\mathrm{RDS}(g) \in \ker(Z_{\mathbf{A}}) \iff Z_{\mathbf{A}}^*(x_1 * x_g) = Z_{\mathbf{A}}^*(x_1)\, Z_{\mathbf{A}}^*(x_g),$$

where $Z_{\mathbf{A}}^* = \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш} \circ \mathrm{q}_G^{-1} : \mathfrak{H}_G^1 \to \mathbf{A}[T]$.

Proof. We present a complete proof for statement (d). The proofs for statements (a), (b), and (c) proceed by similar arguments with appropriate modifications. We have

$$\begin{aligned}
Z_{\mathbf{A}}^*(x_1) &= \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш} \circ \mathrm{q}_G^{-1}(x_1) = T, \\
Z_{\mathbf{A}}^*(x_g) &= \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш} \circ \mathrm{q}_G^{-1}(x_g) = Z_{\mathbf{A}}(x_g), \\
Z_{\mathbf{A}}^*(x_1 * x_g) &= \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш} \circ \mathrm{q}_G^{-1}(x_1 x_g + x_g x_1 + x_0 x_g) = \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш}(x_1 x_g + x_g x_g + x_0 x_g) \\
&= \rho_{Z_{\mathbf{A}}}^{-1} \circ Z_{\mathbf{A}}^{ш}(x_1 ш x_g - x_g x_1 + x_g x_g + x_0 x_g) \\
&= \rho_{Z_{\mathbf{A}}}^{-1}\left(Z_{\mathbf{A}}^{ш}(x_1) Z_{\mathbf{A}}^{ш}(x_g)\right) + Z_{\mathbf{A}}(x_0 x_g + x_g x_g - x_g x_1) \\
&= T\, Z_{\mathbf{A}}(x_g) + Z_{\mathbf{A}}(x_0 x_g + x_g x_g - x_g x_1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
Z_{\mathbf{A}}^*(x_1 * x_g) - Z_{\mathbf{A}}^*(x_1)\, Z_{\mathbf{A}}^*(x_g) &= T\, Z_{\mathbf{A}}(x_g) + Z_{\mathbf{A}}(x_0 x_g + x_g x_g - x_g x_1) - T\, Z_{\mathbf{A}}(x_g) \\
&= Z_{\mathbf{A}}(x_0 x_g + x_g x_g - x_g x_1).
\end{aligned}$$

Hence,

$$Z_{\mathbf{A}}^*(x_1 * x_g) - Z_{\mathbf{A}}^*(x_1)\, Z_{\mathbf{A}}^*(x_g) = Z_{\mathbf{A}}\left(\mathrm{RDS}_{Z_{\mathbf{A}}}(g)\right),$$

thus proving that the vanishing of the left-hand side is equivalent to that of the right-hand side, which is the wanted result. ☐

For any divisor d of the order of G denote

$$K_d \coloneqq \{g \in G | g^d = 1\} = \ker(p^d),$$

which is a subgroup of G of order d.
The following theorem is a generalization of [Zha10, Theorem 4.6].

[5]Here we use the notation RDS similarly to [Zha10], where extended double shuffle relations are called regularized double shuffle relations.

Theorem 2.14. *Let d be a divisor of the order of G. For $h \in G^d$, we have (equality in $\mathfrak{H}_G^0$)*

$$\mathrm{FDT}_{d,1}(h) = \begin{cases} \sum\limits_{g_1,g_2 \in K_d \smallsetminus \{1\}} \mathrm{FDS}(g_1,g_2) + 2 \sum\limits_{g \in K_d \smallsetminus \{1\}} \mathrm{RDS}(g) & \text{if } h = 1 \\ \sum\limits_{\substack{g_1^d = h \\ g_2 \in K_d \smallsetminus \{1\}}} \mathrm{FDS}(g_1,g_2) + \sum\limits_{g^d = h} \mathrm{RDS}(g) - \mathrm{RDS}(h) & \text{otherwise} \\ \qquad + \mathrm{FDT}_{d,2}(h,1) - \mathrm{FDT}_{d,2}(h,h) & \end{cases}$$

Proof. We proceed by considering two distinct cases for h.

Case 1 If $h = 1$, we have

$$\begin{aligned}
&\sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} \mathrm{FDS}(g_1,g_2) + 2 \sum_{g \in K_d \smallsetminus \{1\}} \mathrm{RDS}(g) \\
&= \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} (x_0 x_{g_1 g_2} + x_{g_1} x_{g_1 g_2} + x_{g_2} x_{g_1 g_2} - x_{g_1} x_{g_2} - x_{g_2} x_{g_1}) + 2 \sum_{g \in K_d \smallsetminus \{1\}} (x_0 x_g + x_g x_g - x_g x_1) \\
&= \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} x_0 x_{g_1 g_2} + \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} x_{g_1} x_{g_1 g_2} + \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} x_{g_1} x_{g_1 g_2} - \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} x_{g_1} x_{g_2} \\
&- \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} x_{g_1} x_{g_2} + 2 \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_g + 2 \sum_{g \in K_d \smallsetminus \{1\}} x_g x_g - 2 \sum_{g \in K_d \smallsetminus \{1\}} x_g x_1 \\
&= \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} x_0 x_{g_1 g_2} + 2 \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} x_{g_1} x_{g_1 g_2} - 2 \sum_{g_1,g_2 \in K_d \smallsetminus \{1\}} x_{g_1} x_{g_2} + 2 \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_g \\
&+ 2 \sum_{g \in K_d \smallsetminus \{1\}} x_g x_g - 2 \sum_{g \in K_d \smallsetminus \{1\}} x_g x_1 \\
&= \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_{g^2} + \sum_{\substack{g_1,g_2 \in K_d \smallsetminus \{1\} \\ g_2 \neq g_1}} x_0 x_{g_1 g_2} + 2 \sum_{g \in K_d \smallsetminus \{1\}} x_g x_1 + 2 \sum_{\substack{g_1,g_2 \in K_d \smallsetminus \{1\} \\ g_2 \neq g_1^{-1}}} x_{g_1} x_{g_1 g_2} \\
&- 2 \sum_{g \in K_d \smallsetminus \{1\}} x_g x_g - 2 \sum_{\substack{g_1,g_2 \in K_d \smallsetminus \{1\} \\ g_2 \neq g_1}} x_{g_1} x_{g_2} + 2 \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_g + 2 \sum_{g \in K_d \smallsetminus \{1\}} x_g x_g - 2 \sum_{g \in K_d \smallsetminus \{1\}} x_g x_1 \\
&= \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_{g^2} + \sum_{\substack{g_1,g_2 \in K_d \smallsetminus \{1\} \\ g_2 \neq g_1}} x_0 x_{g_1 g_2} + 2 \sum_{\substack{g_1,g_2 \in K_d \smallsetminus \{1\} \\ g_2 \neq g_1^{-1}}} x_{g_1} x_{g_1 g_2} - 2 \sum_{\substack{g_1,g_2 \in K_d \smallsetminus \{1\} \\ g_2 \neq g_1}} x_{g_1} x_{g_2} \\
&+ 2 \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_g \\
&= \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_{g^2} + \sum_{g_1 \in K_d \smallsetminus \{1\}} \Big(\sum_{g_2 \in K_d} x_0 x_{g_2} - x_0 x_{g_1^2} - x_0 x_{g_1} \Big) + 2 \sum_{\substack{g_1,g_2 \in K_d \smallsetminus \{1\} \\ g_2 \neq g_1}} x_{g_1} x_{g_2} \\
&- 2 \sum_{\substack{g_1,g_2 \in K_d \smallsetminus \{1\} \\ g_2 \neq g_1}} x_{g_1} x_{g_2} + 2 \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_g \\
&= \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_{g^2} + (d-1) \sum_{g \in K_d} x_0 x_g - \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_{g^2} - \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_g + 2 \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_g \\
&= (d-1) \sum_{g \in K_d} x_0 x_g + \sum_{g \in K_d \smallsetminus \{1\}} x_0 x_g = (d-1) \sum_{g \in K_d} x_0 x_g + \sum_{g \in K_d} x_0 x_g - x_0 x_1
\end{aligned}$$

$$= d\sum_{g\in K_d} x_0x_g - x_0x_1 = \mathrm{FDT}_{d,1}(1),$$

where the sixth equality comes from

$$\sum_{g_2\in K_d} x_0x_{g_1g_2} = \sum_{g_2\in K_d} x_0x_{g_2}$$

for any $g_1 \in K_d$, and from

$$\sum_{\substack{g_1,g_2\in K_d\smallsetminus\{1\}\\ g_2\neq g_1^{-1}}} x_{g_1}x_{g_1g_2} = \sum_{\substack{g_1,g_2\in K_d\smallsetminus\{1\}\\ g_2\neq g_1}} x_{g_1}x_{g_2}.$$

Case 2 If $h \neq 1$, we have

$$\sum_{\substack{g_1^d=h\\ g_2\in K_d\smallsetminus\{1\}}} \mathrm{FDS}(g_1,g_2) + \sum_{g^d=h}\mathrm{RDS}(g) - \mathrm{RDS}(h) + \mathrm{FDT}_{d,2}(h,1) - \mathrm{FDT}_{d,2}(h,h)$$
$$= \sum_{\substack{g_1^d=h\\ g_2\in K_d\smallsetminus\{1\}}} (x_0x_{g_1g_2} + x_{g_1}x_{g_1g_2} + x_{g_2}x_{g_1g_2} - x_{g_1}x_{g_2} - x_{g_2}x_{g_1}) + \sum_{g^d=h}(x_0x_g + x_gx_g - x_gx_1)$$
$$- (x_0x_h + x_hx_h - x_hx_1) + \Big(\sum_{\substack{g_1^d=h\\ g_2\in K_d}} x_{g_1}x_{g_2} - x_hx_1\Big) - \Big(\sum_{g_1^d=g_2^d=h} x_{g_1}x_{g_2} - x_hx_h\Big)$$
$$= \sum_{g_2\in K_d\smallsetminus\{1\}}\sum_{g_1^d=h} x_0x_{g_1g_2} + \sum_{g_1^d=h}\sum_{g_2\in K_d\smallsetminus\{1\}} x_{g_1}x_{g_1g_2} + \sum_{g_2\in K_d\smallsetminus\{1\}}\sum_{g_1^d=h} x_{g_2}x_{g_1g_2}$$
$$- \sum_{g_1^d=h}\sum_{g_2\in K_d\smallsetminus\{1\}} x_{g_1}x_{g_2} - \sum_{g_1^d=h}\sum_{g_2\in K_d\smallsetminus\{1\}} x_{g_2}x_{g_1} + \sum_{g^d=h} x_0x_g + \sum_{g^d=h} x_gx_g - \sum_{g^d=h} x_gx_1$$
$$- x_0x_h + \sum_{g_1^d=h}\sum_{g_2\in K_d} x_{g_1}x_{g_2} - \sum_{g_1^d=h}\sum_{g_2^d=h} x_{g_1}x_{g_2}$$
$$= \sum_{g_2\in K_d\smallsetminus\{1\}}\sum_{g_1^d=h} x_0x_{g_1} + \sum_{g_1^d=h}\sum_{\substack{g_2^d=h\\ g_2\neq g_1}} x_{g_1}x_{g_2} + \sum_{g_2\in K_d\smallsetminus\{1\}}\sum_{g_1^d=h} x_{g_2}x_{g_1}$$
$$- \sum_{g_1^d=h}\sum_{g_2\in K_d\smallsetminus\{1\}} x_{g_1}x_{g_2} - \sum_{g_1^d=h}\sum_{g_2\in K_d\smallsetminus\{1\}} x_{g_2}x_{g_1} + \sum_{g^d=h} x_0x_g + \sum_{g^d=h} x_gx_g - \sum_{g^d=h} x_gx_1$$
$$- x_0x_h + \sum_{g_1^d=h}\sum_{g_2\in K_d} x_{g_1}x_{g_2} - \sum_{g_1^d=h}\sum_{g_2^d=h} x_{g_1}x_{g_2}$$
$$= (d-1)\sum_{g^d=h} x_0x_g - \sum_{g^d=h} x_gx_g + \sum_{g^d=h} x_gx_1 + \sum_{g^d=h} x_0x_g + \sum_{g^d=h} x_gx_g - \sum_{g^d=h} x_gx_1 - x_0x_h$$
$$= d\sum_{g^d=h} x_0x_g - x_0x_h = \mathrm{FDT}_{d,1}(h),$$

where the third equality follows from

$$\sum_{g_1^d=h} x_0x_{g_1g_2} = \sum_{g_1^d=h} x_0x_{g_1},$$

for any $g_2 \in K_d$; and from

$$\sum_{g_2 \in K_d} x_{g_1} x_{g_1 g_2} = \sum_{g_2^d = h} x_{g_1} x_{g_2},$$

for any g_1 such that $g_1^d = h$; and from

$$\sum_{g_1^d = h} x_{g_2} x_{g_1 g_2} = \sum_{g_1^d = h} x_{g_2} x_{g_1},$$

for any $g_2 \in K_d$.

□

Proof of Theorem 2.10. Depending of the values of h_1 and h_2, we summarize all possible cases in the following table:

h_1 \ h_2	0	1	$G^d \smallsetminus \{1\}$
0	**Case 3**	**Case 4**	**Case 5**
1	**Case 6**	**Case 2**	**Case 7**
$G^d \smallsetminus \{1\}$	**Case 6**	**Case 1**	**Case 1**

Case 1 $h_1 \in G^d \smallsetminus \{1\}$ and $h_2 \in G^d$. This follows immediately from assumption (iii).
Case 2 $h_1 = h_2 = 1$. This follows from **Step 1** of the proof of Proposition 2.7 with $m = 2$.
Case 3 $h_1 = h_2 = 0$. We have

$$\overline{Z}^{ш}_{\mathbf{A}} \circ p^d_{\sharp}(x_0) = d\overline{Z}^{ш}_{\mathbf{A}}(x_0) = 0. \tag{24}$$

Then

$$\overline{Z}^{ш}_{\mathbf{A}} \circ p^d_{\sharp}(x_0^2) = \overline{Z}^{ш}_{\mathbf{A}} \circ p^d_{\sharp}\left(\frac{1}{2} x_0 \,ш\, x_0\right) = \frac{1}{2}\overline{Z}^{ш}_{\mathbf{A}} \circ p^d_{\sharp}(x_0)^2 = 0,$$

where the second equality follows from the fact that $\overline{Z}^{ш}_{\mathbf{A}} \circ p^d_{\sharp} : \mathfrak{H}^0 G^d, ш \to \mathbf{A}[T]$ is an algebra homomorphism and the last one from identity (24). On the other hand,

$$\overline{Z}^{ш}_{\mathbf{A}} \circ i^{\sharp}_d(x_0) = \overline{Z}^{ш}_{\mathbf{A}}(x_0) = 0. \tag{25}$$

Then

$$\sigma_{Z_{\mathbf{A}}} \circ \overline{Z}^{ш}_{\mathbf{A}} \circ i^{\sharp}_d(x_0^2) = \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}^{ш}_{\mathbf{A}} \circ i^{\sharp}_d\left(\frac{1}{2} x_0 \,ш\, x_0\right) = \frac{1}{2}\sigma_{Z_{\mathbf{A}}}\left(\overline{Z}^{ш}_{\mathbf{A}} \circ i^{\sharp}_d(x_0)^2\right) = 0,$$

where the second equality follows from the fact that $\overline{Z}^{ш}_{\mathbf{A}} \circ i^{\sharp}_d : \mathfrak{H}^0 G^d, ш \to \mathbf{A}[T]$ is an algebra homomorphism and the last one from identity (25).
Case 4 $h_1 = 0$ and $h_2 = 1$. From Theorem 2.14, we have that

$$Z_{\mathbf{A}}\left(\mathrm{FDT}_{Z_{\mathbf{A}},d,1}(1)\right) = \sum_{g_1, g_2 \in K_d \smallsetminus \{1\}} Z_{\mathbf{A}}\left(\mathrm{FDS}_{Z_{\mathbf{A}}}(g_1, g_2)\right) + 2 \sum_{g \in K_d \smallsetminus \{1\}} Z_{\mathbf{A}}\left(\mathrm{RDS}_{Z_{\mathbf{A}}}(g)\right). \tag{26}$$

From assumption (i), one may apply Lemma 2.13 (c) and (d) to conclude that the right hand side of equality (26) is equal to zero. The result then follows by applying Lemma 2.13 (a).
Case 5 $h_1 = 0$ and $h_2 = h \in G^d \smallsetminus \{1\}$. From Theorem 2.14, we have that

$$\begin{aligned} Z_{\mathbf{A}}\left(\mathrm{FDT}_{Z_{\mathbf{A}},d,1}(h)\right) = & \sum_{\substack{g_1^d = h \\ g_2 \in K_d \smallsetminus \{1\}}} Z_{\mathbf{A}}\left(\mathrm{FDS}_{Z_{\mathbf{A}}}(g_1, g_2)\right) + \sum_{g^d = h} Z_{\mathbf{A}}\left(\mathrm{RDS}_{Z_{\mathbf{A}}}(g)\right) \\ & - Z_{\mathbf{A}}\left(\mathrm{RDS}_{Z_{\mathbf{A}}}(h)\right) + Z_{\mathbf{A}}\left(\mathrm{FDT}_{Z_{\mathbf{A}},d,2}(h,1)\right) - Z_{\mathbf{A}}\left(\mathrm{FDT}_{Z_{\mathbf{A}},d,2}(h,h)\right). \end{aligned} \tag{27}$$

From assumption (i) (resp. (iii)), one may apply Lemma 2.13 (c) and (d) (resp. (b)) to conclude that the right hand side of equality (27) is equal to zero. The result then follows by applying Lemma 2.13 (a).

Case 6 $h_1 = h \in G^d$, $h_2 = 0$. We have

$$\begin{aligned}\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_h x_0) &= \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_h \text{ ш } x_0 - x_0 x_h) = \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_h \text{ ш } x_0) - \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_0 x_h)\\ &= \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_h)\, \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_0) - Z_{\mathbf{A}} \circ p^d_\sharp(x_0 x_h) = -Z_{\mathbf{A}} \circ p^d_\sharp(x_0 x_h)\\ &= -Z_{\mathbf{A}} \circ i^\sharp_d(x_0 x_h),\end{aligned}$$

where the second to last equality follows from equality (24) and the last one from the identity obtained in **Case 4** ($h = 1$) and **Case 5** ($h \neq 1$). On the other hand,

$$\begin{aligned}\sigma_{Z_{\mathbf{A}}} \circ \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_h x_0) &= \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_h \text{ ш } x_0 - x_0 x_h)\\ &= \sigma_{Z_{\mathbf{A}}} \left(\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_h \text{ ш } x_0) - \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_0 x_h) \right)\\ &= \sigma_{Z_{\mathbf{A}}} \left(\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_h)\, \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_0) - Z^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_0 x_h) \right)\\ &= -Z_{\mathbf{A}} \circ i^\sharp_d(x_0 x_h),\end{aligned}$$

where the second to last equality follows from equality (25) and the last one from the fact that $x_0 x_h \in \mathfrak{H}^0_{G^d}$ and $\sigma_{Z_{\mathbf{A}}}(1) = 1$.

Case 7 $h_1 = 1$, $h_2 = h \in G^d \smallsetminus \{1\}$. We have

$$\begin{aligned}\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_1 x_h) &= \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_1 \text{ ш } x_h - x_h x_1) = \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_1 \text{ ш } x_h) - \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_h x_1)\\ &= \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_1)\, \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_h) - Z_{\mathbf{A}} \circ p^d_\sharp(x_h x_1)\\ &= \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_1)\, Z^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp(x_h) - Z_{\mathbf{A}} \circ i^\sharp_d(x_h x_1)\\ &= \sigma_{Z_{\mathbf{A}}} \left(\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_1) \right) Z^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_h) - \sigma_{Z_{\mathbf{A}}} \left(Z_{\mathbf{A}} \circ i^\sharp_d(x_h x_1) \right)\\ &= \sigma_{Z_{\mathbf{A}}} \left(\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_1)\, Z^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_h) - Z_{\mathbf{A}} \circ i^\sharp_d(x_h x_1) \right)\\ &= \sigma_{Z_{\mathbf{A}}} \left(\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_1 \text{ ш } x_h - x_h x_1) \right) = \sigma_{Z_{\mathbf{A}}} \circ \overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d(x_1 x_h),\end{aligned}$$

where the third equality follows from the fact that $\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ p^d_\sharp : \mathfrak{H}^0_{G^d,\text{ш}} \to \mathbf{A}[T]$ is an algebra homomorphism; the fourth one from **Step 1** of the proof of Proposition 2.7 and assumption (iii), the fifth one from assumption (ii) and the fact that $\sigma_{Z_{\mathbf{A}}}(1) = 1$, the sixth one from **A**-linearity of $\sigma_{Z_{\mathbf{A}}}$ and the seventh one from the fact that $\overline{Z}^{\text{ш}}_{\mathbf{A}} \circ i^\sharp_d : \mathfrak{H}^0_{G^d,\text{ш}} \to \mathbf{A}[T]$ is an algebra homomorphism.

□

Remark 2.15. Applying the reasoning analogous to that in Proposition 2.11, one can verify that the statement in **Case 7** is equivalent to [Zha10, Conjecture 7.1]. Consequently, the proof provided in **Case 7** also establishes this conjecture.

Appendix A. Quasi-shuffle products

We will consider now a more general point of view for all the products considered before. For this, suppose that we have a set $\mathcal{L}$, called the set of *letters*, and a commutative and associative product $\diamond$ in $\mathcal{L}$. Extending this product bi-linearly to $\mathbb{Q}\mathcal{L}$ gives a commutative non-unital $\mathbb{Q}$-algebra $(\mathbb{Q}\mathcal{L}, \diamond)$. We will be interested in $\mathbb{Q}$-linear combinations of multiple letters of $\mathcal{L}$, i.e., in elements of $\mathbb{Q}\langle\mathcal{L}\rangle$. Here and in the following we call monic monomials in $\mathbb{Q}\langle\mathcal{L}\rangle$ *words*. Moreover, we call the degree of this monomial the *length* of the word.

Definition A.1. Define the *quasi-shuffle product* $*_\diamond$ on $\mathbb{Q}\langle\mathcal{L}\rangle$ as the $\mathbb{Q}$-bilinear product which satisfies $1 *_\diamond w = w *_\diamond 1 = w$ for any word $w \in \mathbb{Q}\langle\mathcal{L}\rangle$ and

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a \diamond b)(w *_\diamond v)$$

for any letters $a, b \in \mathcal{L}$ and words $w, v \in \mathbb{Q}\langle\mathcal{L}\rangle$.

This gives a commutative $\mathbb{Q}$-algebra $(\mathbb{Q}\langle\mathcal{L}\rangle, *_\diamond)$ as shown in [HI17]. Moreover, one can equip this algebra with the structure of a Hopf algebra [HI17, Section 3], where the coproduct is given for $w \in \mathbb{Q}\langle\mathcal{L}\rangle$ by the deconcatenation coproduct

$$\Delta(w) = \sum_{uv=w} u \otimes v\,.$$

Now we consider the completed dual Hopf algebra of $(\mathbb{Q}\langle\mathcal{L}\rangle, *_\diamond, \Delta)$. For this, we need to give a restriction on $\diamond$, that is, we assume that any $a \in \mathcal{L}$ just appears in $b \diamond c$ for finitely many $b, c \in \mathcal{L}$. More precisely, define for $a, b, c \in \mathcal{L}$ the coefficients $\lambda^a_{b,c} \in \mathbb{Q}$ by

$$b \diamond c = \sum_{a \in \mathcal{L}} \lambda^a_{b,c} a.$$

In the following, we will assume that for any $a \in \mathcal{L}$ there are just finitely many $b, c \in \mathcal{L}$ with $\lambda^a_{b,c} \neq 0$.

As a consequence of Lemma A.3 below, the dual completed Hopf algebra of $(\mathbb{Q}\langle\mathcal{L}\rangle, *_\diamond, \Delta)$ is then given by $(\mathbf{A}\langle\langle\mathcal{L}\rangle\rangle, \mathrm{conc}, \widehat{\Delta}_{*_\diamond})$, where conc is the concatenation product and $\widehat{\Delta}_{*_\diamond} : \mathbf{A}\langle\langle\mathcal{L}\rangle\rangle \to \mathbf{A}\langle\langle\mathcal{L}\rangle\rangle^{\hat{\otimes}2}$ is the unique $\mathbb{Q}$-algebra homomorphism such that

$$\widehat{\Delta}_{*_\diamond}(a) = a \otimes 1 + 1 \otimes a + \sum_{b,c \in \mathcal{L}} \lambda^a_{b,c}\, b \otimes c$$

for any $a \in \mathcal{L}$. Since we just consider $\diamond$ such that for any $a \in \mathcal{L}$ the coefficients $\lambda^a_{b,c}$ are zero for almost all $b, c \in \mathcal{L}$ and therefore $\widehat{\Delta}_{*_\diamond}$ is well-defined.

Definition A.2. Define the pairing

$$\begin{aligned} \mathbf{A}\langle\langle\mathcal{L}\rangle\rangle \otimes \mathbb{Q}\langle\mathcal{L}\rangle &\longrightarrow \mathbf{A} \\ \psi \otimes w &\longmapsto (\psi \mid w), \end{aligned}$$

where, as before, $(\psi \mid w)$ denotes the coefficient of w in ψ.

Lemma A.3. *For any $\Phi \in \mathbf{A}\langle\langle\mathcal{L}\rangle\rangle$ we have*

$$\widehat{\Delta}_{*_\diamond}(\Phi) = \sum_{u,v \in \mathcal{L}^*} (\Phi \mid u *_\diamond v) u \otimes v.$$

Proof. Notice that the statement is equivalent to $(\widehat{\Delta}_{*_\diamond}(\Phi) \mid u \otimes v) = (\Phi \mid u *_\diamond v)$ for all $u, v \in \mathcal{L}^*$. Due to linearity, it suffices to show

$$(\widehat{\Delta}_{*_\diamond}(w) \mid u \otimes v) = (w \mid u *_\diamond v) \tag{28}$$

for words $w, u, v \in \mathcal{L}^*$. We will perform induction on the length of w. In the length case 1, that is, when $w = a \in \mathcal{L}$ we have

$$\widehat{\Delta}_{*_\diamond}(a) = a \otimes 1 + 1 \otimes a + \sum_{a_1, a_2 \in \mathcal{L}} \lambda^a_{a_1,a_2}\, a_1 \otimes a_2.$$

By the definition of $u *_\diamond v$ we see that $u = a, v = 1$ or $u = 1, v = a$ or $u = a_1, v = a_2$ (with multiplicity $\lambda^a_{a_1,a_2}$) are exactly cases where a can appear as a coefficient of $u *_\diamond v$ and therefore $(\widehat{\Delta}_{*_\diamond}(a) \mid u \otimes v) = (a \mid u *_\diamond v)$.

Next, notice that (28) is trivial for any w when u or v is the empty word. Therefore, we can restrict ourselves to the case $w = aw'$, $u = bu'$ and $v = cv'$, for letters $a, b, c \in \mathcal{L}$ and words $w', u', v' \in \mathcal{L}^*$. Then we want to show that

$$(\widehat{\Delta}_{*_\diamond}(aw') \mid bu' \otimes cv') = (aw' \mid bu' *_\diamond cv'). \tag{29}$$

For the left-hand side of (29) we have

$$\begin{aligned}
(\widehat{\Delta}_{*_\diamond}(aw') \mid bu' \otimes cv') &= (\widehat{\Delta}_{*_\diamond}(a)\widehat{\Delta}_{*_\diamond}(w') \mid bu' \otimes cv') \\
&= ((a \otimes 1)\widehat{\Delta}_{*_\diamond}(w') \mid bu' \otimes cv') + ((1 \otimes a)\widehat{\Delta}_{*_\diamond}(w') \mid bu' \otimes c'v) \\
&\quad + \sum_{a_1, a_2 \in \mathcal{L}} \lambda^a_{a_1,a_2} \, ((a_1 \otimes a_2)\widehat{\Delta}_{*_\diamond}(w') \mid bu' \otimes cv') \\
&= \delta_{a,b}(\widehat{\Delta}_{*_\diamond}(w') \mid u' \otimes cv') + \delta_{a,c}(\widehat{\Delta}_{*_\diamond}(w') \mid bu' \otimes v') \\
&\quad + \sum_{a_1, a_2 \in \mathcal{L}} \lambda^a_{a_1,a_2} \delta_{a_1,b}\delta_{a_2,c} \, (\widehat{\Delta}_{*_\diamond}(w') \mid u' \otimes v') \\
&= \delta_{a,b}(\widehat{\Delta}_{*_\diamond}(w') \mid u' \otimes cv') + \delta_{a,c}(\widehat{\Delta}_{*_\diamond}(w') \mid bu' \otimes v') \\
&\quad + \lambda^a_{b,c} \, (\widehat{\Delta}_{*_\diamond}(w') \mid u' \otimes v').
\end{aligned}$$

Here, $\delta_{a,b}$ denotes the Kronecker delta. Using the definition of $bu' *_\diamond cv'$, the right-hand side of (29) is given by

$$\begin{aligned}
(aw' \mid bu' *_\diamond cv') &= (aw' \mid b(u' *_\diamond cv')) + (aw' \mid c(bu' *_\diamond v')) + (aw' \mid (b \diamond c)(u' *_\diamond v')) \\
&= \delta_{a,b}(w' \mid u' *_\diamond cv') + \delta_{a,c}(w' \mid bu' *_\diamond v') + \lambda^a_{b,c}(w' \mid u' *_\diamond v').
\end{aligned}$$

Equation (29) then follows from the induction hypothesis. □

As before, denote the set grouplike elements of $\mathbf{A}\langle\langle \mathcal{L} \rangle\rangle$ for the coproduct $\widehat{\Delta}_{*_\diamond}$ by

$$\mathcal{G}(\mathbf{A}\langle\langle \mathcal{L} \rangle\rangle) = \{\Phi \in \mathbf{A}\langle\langle \mathcal{L} \rangle\rangle^\times \mid \widehat{\Delta}_{*_\diamond}(\Phi) = \Phi \otimes \Phi\}.$$

Given a $\mathbb{Q}$-linear map $\varphi : \mathbb{Q}\langle \mathcal{L} \rangle \to \mathbf{A}$ we define the following element in $\mathbf{A}\langle\langle \mathcal{L} \rangle\rangle$

$$\Phi_\varphi := \sum_{w \in \mathcal{L}^*} \varphi(w) w \,.$$

Proposition A.4. *The following two statements are equivalent*

(a) The map $\varphi : (\mathbb{Q}\langle \mathcal{L} \rangle, *_\diamond) \to \mathbf{A}$ is an $\mathbb{Q}$-algebra homomorphism.
(b) We have $\Phi_\varphi \in \mathcal{G}(\mathbf{A}\langle\langle \mathcal{L} \rangle\rangle)$.

Proof. First notice that Lemma A.3 gives

$$\widehat{\Delta}_{*_\diamond}(\Phi_\varphi) = \sum_{w,v \in \mathcal{L}^*} \underbrace{(\Phi_\varphi | w *_\diamond v)}_{= \varphi(w *_\diamond v)} \, w \otimes v.$$

By direct calculation we have

$$\Phi_\varphi \otimes \Phi_\varphi = \left(\sum_{w \in \mathcal{L}^*} \varphi(w) w \right) \otimes \left(\sum_{v \in \mathcal{L}^*} \varphi(v) v \right) = \sum_{w,v \in \mathcal{L}^*} \varphi(w)\varphi(v) \, w \otimes v.$$

Therefore, $\widehat{\Delta}_{*_\diamond}(\Phi_\varphi) = \Phi_\varphi \otimes \Phi_\varphi$ if and only if $\varphi(w *_\diamond v) = \varphi(w)\varphi(v)$ for all $w, v \in \mathcal{L}^*$. □

Appendix B. Functoriality results

In this appendix, we revisit the functors $(-)^*$ and $(-)_*$ defined in [Rac02]. This overview aims to clarify key aspects and highlight some important properties of these functors. This will provide a useful foundation for the definition of their respective dual counterparts $(-)^\sharp$ and $(-)_\sharp$ that we introduce in this paper.

Throughout this appendix we will denote by FinAb the category of finite abelian groups and $\mathbf{A}$-Alg the category of $\mathbf{A}$-algebras.

Proposition-Definition B.1 and Lemmas B.2 and B.3 below are mentioned in [Rac02, §2.5.3] without proof.

Proposition-Definition B.1 ([Rac02, §2.5.3])**.**

(a) Define the mapping $(-)^* : \mathsf{FinAb} \to \mathbf{A}\text{-}\mathsf{Alg}$ as follows:

(i) To each group G, associate the algebra $\mathbf{A}\langle\langle X_G\rangle\rangle$;

(ii) To each group homomorphism $\phi : G_1 \to G_2$, associate the algebra homomorphism $\phi^* : \mathbf{A}\langle\langle X_{G_2}\rangle\rangle \to \mathbf{A}\langle\langle X_{G_1}\rangle\rangle$ given by

$$x_0 \mapsto x_0; \qquad x_h \mapsto \sum_{g\in\phi^{-1}(\{h\})} x_g \quad (h \in G_2).$$

Then the mapping $(-)^*$ is a contravariant functor.

(b) Define the mapping $(-)_* : \mathsf{FinAb} \to \mathbf{A}\text{-}\mathsf{Alg}$ as follows:

(i) To each group G, associate the algebra $\mathbf{A}\langle\langle X_G\rangle\rangle$;

(ii) To each group homomorphism $\phi : G_1 \to G_2$, associate the algebra homomorphism $\phi_* : \mathbf{A}\langle\langle X_{G_1}\rangle\rangle \to \mathbf{A}\langle\langle X_{G_2}\rangle\rangle$ given by

$$x_0 \mapsto d\, x_0 \quad (d = |\ker(\phi)|); \qquad x_g \mapsto x_{\phi(g)} \quad (g \in G_1).$$

Then the mapping $(-)_*$ is a covariant functor.

Proof.

(a) This can be established through direct verification, utilizing the contravariance of the power set functor P^*. Recall that P^* maps each set E to its power set $\mathcal{P}(E)$ and each map $f : E \to F$ to the map $P^*(f) : \mathcal{P}(F) \to \mathcal{P}(E)$ defined by $P^*(f)(U) = f^{-1}(U)$ for $U \subseteq F$.

(b) This can be established through direct verification, utilizing the covariance of the power set functor P_*. Recall that P_* maps each set E to its power set $\mathcal{P}(E)$ and each map $f : E \to F$ to the map $P_*(f) : \mathcal{P}(E) \to \mathcal{P}(F)$ defined by $P_*(f)(S) = f(S)$ for $S \subseteq E$.

□

Lemma B.2. *Let $\phi : G_1 \to G_2$ be an arrow in FinAb. Then, the algebra homomorphism $\phi^* : \mathbf{A}\langle\langle X_{G_2}\rangle\rangle \to \mathbf{A}\langle\langle X_{G_1}\rangle\rangle$ (resp. $\phi_* : \mathbf{A}\langle\langle X_{G_1}\rangle\rangle \to \mathbf{A}\langle\langle X_{G_2}\rangle\rangle$):*

(a) is a Hopf algebra homomorphism $\phi^* : (\mathbf{A}\langle\langle X_{G_2}\rangle\rangle, \widehat{\Delta}_{G_2,ш}) \to (\mathbf{A}\langle\langle X_{G_1}\rangle\rangle, \widehat{\Delta}_{G_1,ш})$ (resp. $\phi_* : (\mathbf{A}\langle\langle X_{G_1}\rangle\rangle, \widehat{\Delta}_{G_1,ш}) \to (\mathbf{A}\langle\langle X_{G_2}\rangle\rangle, \widehat{\Delta}_{G_2,ш})$);

(b) restricts to an algebra homomorphism $\phi^* : \mathbf{A}\langle\langle Y_{G_2}\rangle\rangle \to \mathbf{A}\langle\langle Y_{G_1}\rangle\rangle$ (resp. $\phi_* : \mathbf{A}\langle\langle Y_{G_1}\rangle\rangle \to \mathbf{A}\langle\langle Y_{G_2}\rangle\rangle$), which is a Hopf algebra homomorphism $(\mathbf{A}\langle\langle Y_{G_2}\rangle\rangle, \widehat{\Delta}_{G_2,*}) \to (\mathbf{A}\langle\langle Y_{G_1}\rangle\rangle, \widehat{\Delta}_{G_1,*})$ (resp. $(\mathbf{A}\langle\langle Y_{G_1}\rangle\rangle, \widehat{\Delta}_{G_1,*}) \to (\mathbf{A}\langle\langle Y_{G_2}\rangle\rangle, \widehat{\Delta}_{G_2,*})$).

Proof.

(a) The fact that $\widehat{\Delta}_{G_1,ш} \circ \phi^* = (\phi^*)^{\otimes 2} \circ \widehat{\Delta}_{G_2,ш}$ (resp. $\widehat{\Delta}_{G_2,ш} \circ \phi_* = (\phi_*)^{\otimes 2} \circ \widehat{\Delta}_{G_1,ш}$) can be established through direct verification on the generators.

(b) This follows from the fact that $\phi^*(x_0^{n-1}x_h) = \sum_{g\in\phi^{-1}(\{h\})} x_0^{n-1}x_g$ for any $(n,h) \in \mathbb{Z}_{>0} \times G_2$ (resp. $\phi_*(x_0^{n-1}x_g) = d^{n-1}x_0^{n-1}x_{\phi(g)}$ for any $(n,g) \in \mathbb{Z}_{>0} \times G_1$). From this, one can establish through a direct verification that $\widehat{\Delta}_{G_1,*} \circ \phi^* = (\phi^*)^{\otimes 2} \circ \widehat{\Delta}_{G_2,*}$ (resp. $\widehat{\Delta}_{G_2,*} \circ \phi_* = (\phi_*)^{\otimes 2} \circ \widehat{\Delta}_{G_1,*}$)

□

Lemma B.3. *Let $\phi : G_1 \to G_2$ be an arrow in* FinAb.

(a) We have an equality of $\mathbf{A}$-module homomorphisms $\mathbf{A}\langle\langle X_{G_2}\rangle\rangle \to \mathbf{A}\langle\langle Y_{G_1}\rangle\rangle$

$$\pi_{Y_{G_1}} \circ \phi^* = \phi^* \circ \pi_{Y_{G_2}}$$

and an equality of $\mathbf{A}$-module homomorphisms $\mathbf{A}\langle\langle X_{G_1}\rangle\rangle \to \mathbf{A}\langle\langle Y_{G_2}\rangle\rangle$

$$\pi_{Y_{G_2}} \circ \phi_* = \phi_* \circ \pi_{Y_{G_1}}.$$

(b) We have an equality of $\mathbf{A}$-module homomorphisms $\mathbf{A}\langle\langle X_{G_2}\rangle\rangle \to \mathbf{A}\langle\langle X_{G_1}\rangle\rangle$

$$\mathbf{q}_{G_1} \circ \phi^* = \phi^* \circ \mathbf{q}_{G_2}$$

and an equality of $\mathbf{A}$-module homomorphisms $\mathbf{A}\langle\langle X_{G_1}\rangle\rangle \to \mathbf{A}\langle\langle X_{G_2}\rangle\rangle$

$$\mathbf{q}_{G_2} \circ \phi_* = \phi_* \circ \mathbf{q}_{G_1}.$$

Proof.

(a) This follows from the direct sum decomposition $\mathbf{A}\langle\langle X_{G_i}\rangle\rangle = \mathbf{A}\langle\langle Y_{G_i}\rangle\rangle \oplus \mathbf{A}\langle\langle X_{G_i}\rangle\rangle x_0$ ($i \in \{1,2\}$) and the fact that ϕ^* (resp. ϕ_*) restricts to $\mathbf{A}\langle\langle Y_{G_2}\rangle\rangle \to \mathbf{A}\langle\langle Y_{G_1}\rangle\rangle$ (resp. $\mathbf{A}\langle\langle Y_{G_1}\rangle\rangle \to \mathbf{A}\langle\langle Y_{G_2}\rangle\rangle$), thanks to Lemma B.2 (b).

(b) It is enough to prove this equality on a word $x_0^{n_1-1}x_{h_1}\cdots x_0^{n_r-1}x_{h_r}x_0^{n_{r+1}-1}$ of $\mathbf{A}\langle\langle X_{G_2}\rangle\rangle$ (resp. $x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}$ of $\mathbf{A}\langle\langle X_{G_1}\rangle\rangle$). We have

$$\begin{aligned}
&\mathbf{q}_{G_1} \circ \phi^*(x_0^{n_1-1}x_{h_1}\cdots x_0^{n_r-1}x_{h_r}x_0^{n_{r+1}-1}) = \sum_{\substack{\phi(g_i)=h_i\\ 1\le i\le r}} \mathbf{q}_{G_1}(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1})\\
&= \sum_{\substack{\phi(g_i)=h_i\\ 1\le i\le r}} x_0^{n_1-1}x_{g_1}x_0^{n_2-1}x_{g_1^{-1}g_2}\cdots x_0^{n_r-1}x_{g_{r-1}^{-1}g_r}x_0^{n_{r+1}-1}\\
&= \sum_{\phi(g_1')=h_1}\ \sum_{\substack{\phi(g_i')=h_{i-1}^{-1}h_i\\ 2\le i\le r}} x_0^{n_1-1}x_{g_1'}x_0^{n_2-1}x_{g_2'}\cdots x_0^{n_r-1}x_{g_r'}x_0^{n_{r+1}-1}\\
&= \phi^*\left(x_0^{n_1-1}x_{h_1}x_0^{n_2-1}x_{h_1^{-1}h_2}\cdots x_0^{n_r-1}x_{h_{r-1}^{-1}h_r}x_0^{n_{r+1}-1}\right)\\
&= \phi^* \circ \mathbf{q}_{G_2}(x_0^{n_1-1}x_{h_1}\cdots x_0^{n_r-1}x_{h_r}x_0^{n_{r+1}-1}),
\end{aligned}$$

where the third equality follows by applying the change of variables

$$g_1' = g_1, \quad g_2' = g_1^{-1}g_2, \quad \dots, \quad g_r' = g_{r-1}^{-1}g_r.$$

For the second identity we get

$$\begin{aligned}
&\mathbf{q}_{G_2} \circ \phi_*(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}) = \mathbf{q}_{G_1}(x_0^{n_1-1}x_{\phi(g_1)}\cdots x_0^{n_r-1}x_{\phi(g_r)}x_0^{n_{r+1}-1})\\
&= x_0^{n_1-1}x_{\phi(g_1)}x_0^{n_2-1}x_{\phi(g_1)^{-1}\phi(g_2)}\cdots x_0^{n_r-1}x_{\phi(g_{r-1})^{-1}\phi(g_r)}x_0^{n_{r+1}-1}\\
&= x_0^{n_1-1}x_{\phi(g_1)}x_0^{n_2-1}x_{\phi(g_1^{-1}g_2)}\cdots x_0^{n_r-1}x_{\phi(g_{r-1}^{-1}g_r)}x_0^{n_{r+1}-1}\\
&= \phi_*\left(x_0^{n_1-1}x_{g_1}x_0^{n_2-1}x_{g_1^{-1}g_2}\cdots x_0^{n_r-1}x_{g_{r-1}^{-1}g_r}x_0^{n_{r+1}-1}\right)\\
&= \phi_* \circ \mathbf{q}_{G_1}(x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}),
\end{aligned}$$

where the third equality follows from the fact that ϕ is a group homomorphism.

□

We now define the dual counterparts of the functors $(-)^*$ and $(-)_*$.

Proposition-Definition B.4.

(a) Define a mapping $(-)^\sharp : \mathsf{FinAb} \to \mathbf{A}\text{-}\mathsf{Alg}$ *as follows:*

(i) To each group G, associate the algebra $\mathfrak{H}_G$;

(ii) To each group homomorphism $\phi : G_1 \to G_2$, associate the algebra homorphism $\phi^\sharp : \mathfrak{H}_{G_1} \to \mathfrak{H}_{G_2}$ given by

$$x_0 \mapsto x_0; \qquad x_g \mapsto x_{\phi(g)} \quad (g \in G_1).$$

Then the mapping $(-)^\sharp$ is a covariant functor.

(b) Define a mapping $(-)_\sharp : \mathsf{FinAb} \to \mathbf{A}\text{-}\mathsf{Alg}$ *as follows:*

(i) To each group G, associate the algebra $\mathfrak{H}_G$;

(ii) To each group homomorphism $\phi : G_1 \to G_2$, associate the algebra homorphism $\phi_\sharp : \mathfrak{H}_{G_2} \to \mathfrak{H}_{G_1}$ given by

$$x_0 \mapsto d\, x_0 \quad (d = |\ker(\phi)|); \qquad x_h \mapsto \sum_{g \in \phi^{-1}(\{h\})} x_g \quad (h \in G_2).$$

Then the mapping $(-)_\sharp$ is a contravariant functor.

Proof. This can be proven by applying the same reasoning used in the proof of Proposition-Definition B.1. □

Let $\phi : G_1 \to G_2$ be an arrow in FinAb. For $i \in \{1, 2\}$, consider the pairing

$$(-,-)_i : \mathbf{A}\langle\langle X_{G_i}\rangle\rangle \otimes \mathfrak{H}_{G_i} \to \mathbf{A},$$

from Definition A.2 for the case $\mathcal{L} = X_{G_i}$.

Lemma B.5. *Let $\phi : G_1 \to G_2$ be an arrow in* FinAb. *We have*

(a) $(\phi^*(S_2), P_1)_1 = (S_2, \phi^\sharp(P_1))_2$, for $S_2 \in \mathbf{A}\langle\langle X_{G_2}\rangle\rangle$ and $P_1 \in \mathfrak{H}_{G_1}$.

(b) $(\phi_*(S_1), P_2)_2 = (S_1, \phi_\sharp(P_2))_1$, for $S_1 \in \mathbf{A}\langle\langle X_{G_1}\rangle\rangle$ and $P_2 \in \mathfrak{H}_{G_2}$.

Proof.

(a) It is enough to show this equality for $P_1 = x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1} \in X_{G_1}^*$, due to linearity. We have $\phi^\sharp(P_1) = x_0^{n_1-1}x_{\phi(g_1)}\cdots x_0^{n_r-1}x_{\phi(g_r)}x_0^{n_{r+1}-1}$. Therefore,

$$\begin{aligned}(\phi^*(S_2)|P_1)_1 &= (\phi^*(S_2)|x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1})_1\\ &= (S_2|x_0^{n_1-1}x_{\phi(g_1)}\cdots x_0^{n_r-1}x_{\phi(g_r)}x_0^{n_{r+1}-1})_2 = (S_2|\phi^\sharp(P_1))_2.\end{aligned}$$

(b) It is enough to show this equality for $P_2 = x_0^{n_1-1}x_{h_1}\cdots x_0^{n_r-1}x_{h_r}x_0^{n_{r+1}-1} \in X_{G_2}^*$, due to linearity. We have

$$\phi_\sharp(P_2) = d^{n_1+\cdots+n_{r+1}-(r+1)} \sum_{\substack{g_i \in \phi^{-1}(\{h_i\})\\ 1\le i\le r}} x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1}.$$

Therefore,

$$\begin{aligned}(\phi_*(S_1)|P_2)_2 &= (\phi_*(S_1)|x_0^{n_1-1}x_{h_1}\cdots x_0^{n_r-1}x_{h_r}x_0^{n_{r+1}-1})_2\\ &= d^{n_1+\cdots+n_{r+1}-(r+1)} \sum_{\substack{g_i \in \phi^{-1}(\{h_i\})\\ 1\le i\le r}} (S_1|x_0^{n_1-1}x_{g_1}\cdots x_0^{n_r-1}x_{g_r}x_0^{n_{r+1}-1})_1\\ &= (S_1|\phi^\sharp(P_2))_1.\end{aligned}$$

□

Corollary B.6. *Let $\phi : G_1 \to G_2$ be an arrow in* FinAb*.*

(a) The map $\phi^\sharp$ (resp. $\phi_\sharp$) is an algebra homomorphism $\mathfrak{H}_{G_1,ш} \to \mathfrak{H}_{G_2,ш}$ (resp. $\mathfrak{H}_{G_2,ш} \to \mathfrak{H}_{G_1,ш}$).

(b) The map $\phi^\sharp$ (resp. $\phi_\sharp$) restricts to an algebra homomorphism $\mathfrak{H}^1_{G_1} \to \mathfrak{H}^1_{G_2}$ (resp. $\mathfrak{H}^1_{G_2} \to \mathfrak{H}^1_{G_1}$). Moreover, it is an algebra homomorphism $\mathfrak{H}^1_{G_1,*} \to \mathfrak{H}^1_{G_2,*}$ (resp. $\mathfrak{H}^1_{G_2,*} \to \mathfrak{H}^1_{G_1,*}$).

Proof.

(a) Let $P, Q \in \mathfrak{H}_{G_1}$. To prove the statement, we will instead demonstrate that for any $w \in X^*_{G_2}$, we have $(\phi^\sharp(P ш Q), w)_2 = (\phi^\sharp(P) ш \phi^\sharp(Q), w)_2$. The equivalence of these statements allows us to proceed as follows:

$$\begin{aligned}(\phi^\sharp(P ш Q), w)_2 &= (P ш Q, \phi^*(w))_1 = \Big(P \otimes Q, \widehat{\Delta}_ш \circ \phi^*(w)\Big)_1 \\ &= \Big(P \otimes Q, (\phi^*)^{\otimes 2} \circ \widehat{\Delta}_ш(w)\Big)_1 = \Big(\phi^\sharp(P) \otimes \phi^\sharp(Q), \widehat{\Delta}_ш(w)\Big)_2 \\ &= \Big(\phi^\sharp(P) ш \phi^\sharp(Q), w\Big)_2,\end{aligned}$$

where the first and fourth equalities come from Lemma B.5, the second and last ones from Lemma A.3 and the third one from Lemma B.2 (a).
The exact same arguments applies for $\phi_\sharp$.

(b) The proof mirrors the reasoning employed in (a), with the key difference being the application of Lemma B.2 (b) instead of B.2 (a).

□

Corollary B.7. *Let $\phi : G_1 \to G_2$ be an arrow in* FinAb*. We have an equality of $\mathbb{Q}$-linear maps $\mathfrak{H}_{G_1} \to \mathfrak{H}_{G_2}$*

$$\mathrm{q}_{G_2} \circ \phi^\sharp = \phi^\sharp \circ \mathrm{q}_{G_1}, \tag{30}$$

and an equality of $\mathbb{Q}$-linear maps $\mathfrak{H}_{G_2} \to \mathfrak{H}_{G_1}$

$$\mathrm{q}_{G_1} \circ \phi_\sharp = \phi_\sharp \circ \mathrm{q}_{G_2}. \tag{31}$$

Proof. Let $P \in \mathfrak{H}_{G_1}$. To prove the statement, we will instead demonstrate that for any $w \in X^*_{G_2}$, we have $(\phi^\sharp \circ \mathrm{q}_{G_1}(P), w)_2 = (\mathrm{q}_{G_2} \circ \phi^\sharp(P), w)_2$. The equivalence of these statements allows us to proceed as follows:

$$\begin{aligned}(\phi^\sharp \circ \mathrm{q}_{G_1}(P), w)_2 &= (\mathrm{q}_{G_1}(P), \phi^*(w))_1 = \Big(P, \mathbf{q}^{-1}_{G_1} \circ \phi^*(w)\Big)_1 = \Big(P, \phi^* \circ \mathbf{q}^{-1}_{G_2}(w)\Big)_1 \\ &= \Big(\phi^\sharp(P), \mathbf{q}^{-1}_{G_2}(w)\Big)_2 = \Big(\mathrm{q}_{G_2} \circ \phi^\sharp(P), w\Big)_2,\end{aligned}$$

where the first and fourth equalities come from Lemma B.5, the second and last ones from identity (5) and the third one from Lemma B.3 (b). Finally, one may apply similar reasoning to prove identity (31). □

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