Max-Planck-Institut für Mathematik

Multiple harmonic q-series (MHS) and finite (FMZV) & symmetrized MZV (SMZV)

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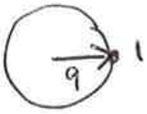
A MHS is a sum of the form

$$\sum_{\mathrm{ord}(q) > m_1 > \dots > m_r > 0} \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}} \qquad Q_j(x) \in \mathbb{Q}[x], \quad k_j \in \mathbb{Z}_{\geq 1}.$$

qMZV
"classical case"
$\tau \in \mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$
$|q| < 1$

$$\text{Mod. forms} \subset \mathrm{MHS} \subset \mathcal{O}(\mathbb{H})$$
$$\downarrow$$
$$\mathrm{MZV}$$



Cusp forms $\rightsquigarrow$ Relations

$q = e^{2\pi i \tau} = e(\tau)$

"cyclotomic case": $\tau \in \mathbb{Q}$, $q^n = 1$ ($n = \mathrm{ord}(q)$)

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$$\begin{tikzcd} & \mathrm{MHS} \subset \mathrm{Map}(\mathbb{Q}, \mathbb{C}) \arrow[dl, "\text{\"q=1\" mod } 1-q"'] \arrow[dr, "\text{\"q}\to\text{1\"}"] & \\ \mathrm{FMZV} \arrow[rr, leftrightarrow, "\text{Kaneko-Zagier}"'] & & \mathrm{SMZV} \end{tikzcd}$$

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1 (S) MZV

$\mathbb{k} = (k_1, \dots, k_r)$, $k_1, \dots, k_r \geq 1$, $\mathrm{wt}(\mathbb{k}) = k_1 + \dots + k_r$

$\mathbb{k}$ adm. $\iff$ $k_1 \geq 2 \vee \mathbb{k} = \emptyset$ $(r=0)$

Def (MZV): For $\mathbb{k}$ adm. define

$$\zeta(\mathbb{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R} \qquad (\zeta(\emptyset) = 1)$$

- MZV satisfy many $\mathbb{Q}$-linear relations, ex: $\zeta(5) = 2\zeta(2,3) - 4\zeta(4,1)$.
- $\mathcal{Z} = \langle \zeta(\mathbb{k}) \mid \mathbb{k} \text{ adm.} \rangle_{\mathbb{Q}}$, $\mathcal{Z}_k = \langle \zeta(\mathbb{k}) \mid \mathbb{k} \text{ adm.}, \mathrm{wt}(\mathbb{k}) = k \rangle_{\mathbb{Q}}$.

Conjecture (Zagier)

$$\sum_{k \geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k X^k = \frac{1}{1 - X^2 - X^3}$$

For any $\mathbb{k}$ let $\zeta(\mathbb{k};T) \in \mathcal{Z}[T]$ denote the stuffle-regularized MZV.

Ex: $\zeta(1,2;T) = \zeta(2)T - 2\zeta(3)$.

Def (SMZV) For any $\mathbb{k}$ define

$$\zeta_S(\mathbb{k}) = \zeta_S(k_1,\dots,k_r) = \sum_{a=0}^{r} (-1)^{k_1+\dots+k_a} \zeta(k_a,\dots,k_1;T)\,\zeta(k_{a+1},\dots,k_r;T) \overset{\text{Prop.}}{\in} \mathcal{Z}$$

depth 1: $k \geq 1$ $\zeta_S(k) = \begin{cases} 2\zeta(k), & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$. $\quad \zeta(3) = \frac{1}{3}\zeta_S(2,1)$

Thm. (Yasuda 2014) $\mathcal{Z} = \langle \zeta_S(\mathbb{k}) \rangle_{\mathbb{Q}}$.

MZV relations $\Longrightarrow$ $\zeta_S(4,1) - \zeta_S(1,4) + \zeta_S(3,2) = 0$

2 FMZV

Def (FMZV) For any $\mathbb{k}$ define

$$\zeta_{\mathcal{A}}(\mathbb{k}) = \zeta_{\mathcal{A}}(k_1,\dots,k_r) = \left(\sum_{p > m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \bmod p \right)_{p \text{ prime}} \in \underbrace{\left(\prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}\right) \Big/ \left(\bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}\right)}_{=:\,\mathcal{A}}$$

- $\mathcal{A}$ is a $\mathbb{Q}$-algebra
- depth 1: $k \geq 1$ $\zeta_{\mathcal{A}}(k) = 0$
- $\mathcal{Z}^{\mathcal{A}} = \langle \zeta_{\mathcal{A}}(\mathbb{k}) \rangle_{\mathbb{Q}}$, $\mathcal{Z}_k^{\mathcal{A}} = \langle \zeta_{\mathcal{A}}(\mathbb{k}) \mid \mathrm{wt}(\mathbb{k}) = k \rangle_{\mathbb{Q}}$
- Satisfy many linear relations, e.g.

$$\zeta_{\mathcal{A}}(4,1) - \zeta_{\mathcal{A}}(1,4) + \zeta_{\mathcal{A}}(3,2) = 0$$

Conjecture (Kaneko-Zagier) We have a $\mathbb{Q}$-algebra isomorphism

$$\mathcal{Z}^{\mathcal{A}} \longrightarrow \mathcal{Z}/\pi^2\mathcal{Z}$$

$$\zeta_{\mathcal{A}}(\mathbb{k}) \longmapsto \zeta_S(\mathbb{k}) \bmod \pi^2\mathcal{Z},$$

$$\sum_{k \geq 0} \dim_{\mathbb{Q}} \mathcal{Z}_k^{\mathcal{A}} X^k = \frac{1-X^2}{1-X^2-X^3}.$$

③ MHS

$$I = \left\{ (\mathbb{k};\mathbb{e}) \;\middle|\; \begin{array}{l} \mathbb{k}=(k_1,\dots,k_r) \quad\quad k_j \geq 1 \\ \mathbb{e}=(e_1,\dots,e_r),\ r\geq 0,\ 0\leq e_j\leq k_j,\ j=1,\dots,r \end{array} \right\}$$

$$\cup$$

$$I^0 = \{ (\mathbb{k};\mathbb{e})\in I \mid 0\leq e_j \leq k_j-1 \ \forall j=1,\dots,r \}.$$

$$I_k = \{ (\mathbb{k};\mathbb{e}) \in I \mid \mathrm{wt}(\mathbb{k}) = k \},\quad I_k^0 = I^0 \cap I_k.$$

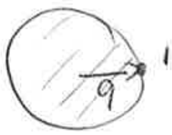
<u>Def</u> (MHS) For $\tau \in \mathbb{Q} \cup \mathbb{H}$ and $(\mathbb{k},\mathbb{e}) \in I$ $\left(\begin{smallmatrix} e_1 \geq 1 \text{ if} \\ \tau\in\mathbb{H}\end{smallmatrix}\right)$ define

$q = e^{2\pi i \tau} = e(\tau)$

$$g(\mathbb{k};\mathbb{e};q) = g\left(\begin{matrix}k_1,\dots,k_r \\ e_1,\dots,e_r\end{matrix};q\right) = \sum_{\mathrm{ord}(q) > m_1 > \dots > m_r > 0} \frac{q^{m_1 e_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r e_r}}{(1-q^{m_r})^{k_r}} \in \mathbb{C}.$$

$$\mathrm{ord}(q) = \begin{cases} \infty & ,\ \tau\in\mathbb{H} \\ \min\{n \mid q^n = 1\} & ,\ \tau \in \mathbb{Q} \end{cases}$$

"classical case": $\tau \in \mathbb{H},\ |q|<1$



- $k_1 \geq 2$: $\lim_{q\to 1} (1-q)^{\mathrm{wt}(\mathbb{k})} g(\mathbb{k};\mathbb{e};q) = \zeta(\mathbb{k})$

- $\sum_{(\mathbb{k},\mathbb{e})\in I_k} \alpha_{\mathbb{k},\mathbb{e}}\, g(\mathbb{k};\mathbb{e};q) \in S_k \Rightarrow \sum_{(\mathbb{k},\mathbb{e})\in I_k} \alpha_{\mathbb{k},\mathbb{e}}\, \zeta(\mathbb{k}) = 0$, with $\alpha_{\mathbb{k},\mathbb{e}} \in \mathbb{Q}$

  S_k: Cusp forms of wt k

- Have - dimension conjectures
  - description of alg structure
  - (conjectured) understanding of linear relations.

"cyclotomic case": From now on $\tau \in \mathbb{Q}$, q root of unity
$(\mathbf{k},\mathbf{e}) \in I$, $g(\mathbf{k};\mathbf{e}) \in \mathrm{Map}(\mathbb{Q},\mathbb{C})$.

$$\mathcal{G}^\circ = \langle g(\mathbf{k};\mathbf{e}) \mid (\mathbf{k},\mathbf{e}) \in I^\circ \rangle_{\mathbb{Q}} .$$

Thm I (BTT) For $(\mathbf{k},\mathbf{e}) \in I^\circ$ we have

i) $\xi(\mathbf{k}) := \lim\limits_{n\to\infty} (1-e(\tfrac{1}{n}))^{\mathrm{wt}(\mathbf{k})} g(\mathbf{k};\mathbf{e};e(\tfrac{1}{n})) = \sum\limits_{a=0}^{r} (-1)^{k_1+\cdots+k_a} \zeta(k_a,\ldots,k_1;\tfrac{\pi i}{2})\,\zeta(k_{a+1},\ldots,k_r;-\tfrac{\pi i}{2})$
$\in \mathbb{Z}+\pi i\mathbb{Z}$.

ii) $\mathrm{Re}(\xi(\mathbf{k})) \equiv \zeta_S(\mathbf{k}) \mod \pi^2\mathbb{Z}$.

$\left(\mathrm{Re}(\xi(\mathbf{k})) = \zeta_S(\mathbf{k}) \text{ if } \begin{array}{l} k_1,\ldots,k_r \geq 2 \\ r=1 \\ r=2,\ \mathbf{k}\neq(1,1) \end{array} \right)$.

- Thm I gives a map $\varphi^S : \mathcal{G}^\circ \longrightarrow \mathcal{Z}$
$\qquad g(\mathbf{k};\mathbf{e}) \longmapsto \mathrm{Re}(\xi(\mathbf{k}))$.

Thm II (BTT) For $(\mathbf{k},\mathbf{e}) \in I^\circ$, p prime, $I_p = (1-e(\tfrac{1}{p})) \subset \mathbb{Z}[e(\tfrac{1}{p})]$ we have

i) $(1-e(\tfrac{1}{p}))^{\mathrm{wt}(\mathbf{k})} g(\mathbf{k};\mathbf{e};e(\tfrac{1}{p})) \in \mathbb{Z}[e(\tfrac{1}{p})]$,

ii) $\mathbb{Z}[e(\tfrac{1}{p})]/I_p \cong \mathbb{Z}/p\mathbb{Z}$,

iii) $\zeta_A(\mathbf{k}) = \left((1-e(\tfrac{1}{p}))^{\mathrm{wt}(\mathbf{k})} g(\mathbf{k};\mathbf{e};e(\tfrac{1}{p})) \mod I_p \right)_{p \text{ prime}}$.

- Thm II gives a map $\varphi^A : \mathcal{G}^\circ \longrightarrow \mathcal{Z}^A$

- In $\mathcal{G}^\circ$ we can prove various relations, e.g.

$$g\binom{4,1}{0,0} - g\binom{1,4}{0,3} + \frac{1}{3}g\binom{3,2}{2,0} + \frac{2}{3}g\binom{3,2}{1,1} - \frac{1}{3}g\binom{3,2}{1,0} + \frac{4}{3}g\binom{3,2}{0,1} - g\binom{3,2}{0,0} = 0$$

$\swarrow \varphi^S \qquad\qquad \searrow \varphi^A$

$\zeta_S(4,1) - \zeta_S(1,4) + \zeta_S(3,2) = 0 \qquad \zeta_A(4,1) - \zeta_A(1,4) + \zeta_A(3,2) = 0$.

Q: What are the kernels of $\varphi^S : \mathcal{G}^\circ \longrightarrow \mathcal{Z} \longrightarrow \mathcal{Z}/\pi^2\mathcal{Z}$?

(Approach: Consider a larger space.) $\varphi^A : \mathcal{G}^\circ \longrightarrow \mathcal{Z}^A$

<u>Thm II'</u>: Thm II is also true for $(\mathbf{k},\mathbf{e}) \in I$.

$$\mathcal{G} = \langle g(\mathbf{k};\mathbf{e}) \mid (\mathbf{k};\mathbf{e}) \in I \rangle_{\mathbb{Q}} \xrightarrow{\varphi^A} \mathcal{Z}^A .$$

But: Thm I does not work for every $(\mathbf{k},\mathbf{e}) \in I$!

Ex: $(\underset{(1,1)}{\mathbf{k}};\underset{(1,0)}{\mathbf{e}}) = (\underset{1,1}{\cancel{\;\;}}; 1,0) \in I \setminus I^\circ$

$$(1-e(\tfrac{1}{n}))^2 g\left(\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}; e(\tfrac{1}{n})\right) = 2\mathcal{G}(2) + 2\pi i \left(\log\left(\tfrac{n}{2\pi}\right) - \gamma\right) + O\left(\tfrac{\log(n)}{n}\right)$$

as $n \to \infty$.

(i.e. $\varphi^S(\mathbf{k};\mathbf{e})$ doesn't exist for all $(\mathbf{k},\mathbf{e}) \in I$)

Now define

$$\widetilde{I} = I \setminus \{ ((\ldots,1,1,\ldots);(\ldots,1,0,\ldots)) , (\ldots,1,1,\ldots;\ldots,0,1,\ldots) \}$$

$$\widetilde{\mathcal{G}} = \langle g(\mathbf{k};\mathbf{e}) \mid (\mathbf{k},\mathbf{e}) \in \widetilde{I} \rangle_{\mathbb{Q}} . \qquad \mathcal{G}^\circ \subset \widetilde{\mathcal{G}} \subset \mathcal{G}$$

<u>Observation I</u>: Can extend φ^S to $\widetilde{\mathcal{G}} \longrightarrow \mathcal{Z}$.

- Denote by $\mathcal{G}^\circ_k$, $\widetilde{\mathcal{G}}_k$ and $\mathcal{G}_k$ the wt k parts.
- The image of φ^A and φ^S is independent of $\mathbf{e}$, so we define:

$$\mathcal{E}_k = \langle g(\mathbf{k};\mathbf{e}_1) - g(\mathbf{k};\mathbf{e}_2) \mid (\mathbf{k};\mathbf{e}_1), (\mathbf{k};\mathbf{e}_2) \in I_k \rangle_{\mathbb{Q}}$$
$$\subset \ker \varphi^A$$

similarly $\mathcal{E}^\circ_k \subset \ker\varphi^A$, $\widetilde{\mathcal{E}}_k \subset \ker\varphi^A$.
$\phantom{similarly\ \mathcal{E}^\circ_k} \subset \ker\varphi^S \qquad \underset{?}{\subset} \ker\varphi^S$

$$\mathcal{Z}_k^{q,0} = \mathfrak{g}_k^0 / \mathcal{E}_k^0, \quad \widetilde{\mathcal{Z}}^q = \widetilde{\mathfrak{g}}_k / \widetilde{\mathcal{E}}_k, \quad \mathcal{Z}_k^q = \mathfrak{g}_k / \mathcal{E}_k .$$

Numerical data:

k	1	2	3	4	5	6	7	8
$\# I^0$	1	3	8	21	55	144	377	987
$\# \widetilde{I}$	2	5	18	56	174	548	1720	5396
$\# I$	2	7	24	82	280	956	3264	11144
$\dim_{\mathbb{Q}} \mathfrak{g}_k^0 \overset{?}{=}$	1	2	4	6	10	15	23	
$\dim_{\mathbb{Q}} \widetilde{\mathfrak{g}}_k \overset{?}{=}$	1	2	5	10	19	34		...
$\dim_{\mathbb{Q}} \mathfrak{g}_k \overset{?}{=}$	1	3	7	13	25	45		
$\dim_{\mathbb{Q}} \mathcal{Z}_k^{q,0} \overset{?}{=}$	1	1	2	2	4	5	8	12
$\dim_{\mathbb{Q}} \widetilde{\mathcal{Z}}_k^q$	0	1	1	1	2	2	3	4
$\dim_{\mathbb{Q}} \mathcal{Z}_k^q$	0	0	1	0	1	1	1	2

Observation II:

- $\dim_{\mathbb{Q}} \widetilde{\mathcal{Z}}_k^q \overset{?}{=} \dim_{\mathbb{Q}} \mathcal{Z}_k$
- $\dim_{\mathbb{Q}} \mathcal{Z}_k^q \overset{?}{=} \dim_{\mathbb{Q}} \mathcal{Z}_k^A$
- $\ker \varphi^A \overset{?}{=} \mathcal{E}_k$.

Conjectural picture:

$i: \tilde{\mathcal{G}}_K/\hat{\mathcal{E}}_K \hookrightarrow \mathcal{G}_K/\mathcal{E}_K$

$$\begin{tikzcd}
 & \tilde{\mathcal{G}} \arrow[d, "\pi"] & g\binom{2}{0} \arrow[d, mapsto] & \\
 & \tilde{\mathcal{Z}}^q_K \arrow[dl, "i"'] \arrow[dr, "\varphi^S", "\sim"'] & \tilde{\zeta}(2) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
\mathcal{Z}^q_K \arrow[d, "\varphi^A", "\wr"'] & 0 \arrow[d, mapsto] & \mathcal{Z}_K \arrow[d, "\pi"] & 2\zeta(2) \arrow[d, mapsto] \\
\mathcal{Z}^A_K & 0 & (\mathcal{Z}/\pi^2\mathcal{Z})_K & 0
\end{tikzcd}$$

Q: $\ker(\varphi^A \circ i) \stackrel{?}{=} \ker(\pi \circ \varphi^S)$.

Ex:

$$g\binom{2}{0} + g\binom{2}{1} = -g\begin{pmatrix}1&1\\0&0\end{pmatrix}$$

$$g\binom{2}{0} + g\binom{2}{1} = g\begin{pmatrix}1&1\\0&1\end{pmatrix} + g\begin{pmatrix}1&1\\1&0\end{pmatrix}$$

$$\Rightarrow \quad g\binom{2}{0} \equiv g\begin{pmatrix}1&1\\0&0\end{pmatrix} \equiv 0 \mod \mathcal{E}_2$$

(but $\not\equiv 0 \mod \hat{\mathcal{E}}_2$.)