Derivations for quasi-shuffle algebras

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[BIM], arxiv: 2312.04124

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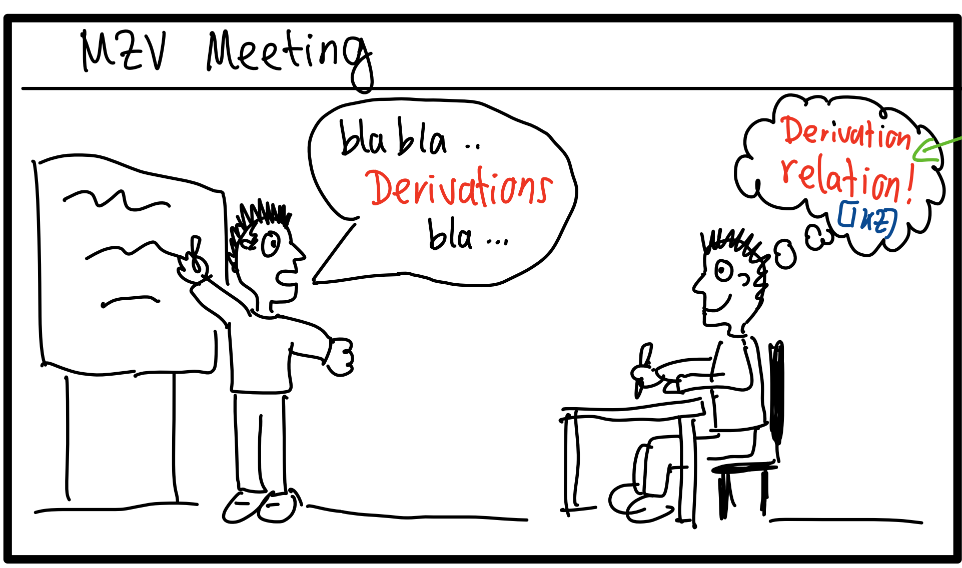
① Derivations

② Quasi-shuffle algebras (QSA)

③ = ① + ②

④ Motivation

Common picture when "derivations" appear at MZV Meeting:



Not related to todays derivations

(just indirectly)

① Derivations

english field
$\#$
german: Körper = "body"
$\|$
Japanese 体 (たい)

A : K- module (usually $K = \mathbb{Q}$)

$A = (A, \odot)$ K- algebra (most of the time, but not always, unitary & commutative)

Derivation on A:

- $d : A \to A$ K- linear
- $d(a \odot b) = d(a) \odot b + a \odot d(b)$ (Leibniz's law)

$\Longrightarrow$ d uniquely determined by its values on the alg. gen. of A.

$\mathrm{Der}(A)$: all derivations on A

$\uparrow$

Lie algebra with $[d_1, d_2] = d_1 \circ d_2 - d_2 \circ d_1$.

Assume A is graded $:\Leftrightarrow$ $A = \bigoplus_{k \geq 0} A_k$

$A_k \odot A_\ell \subset A_{k+\ell}$

We say $d \in \mathrm{Der}(A)$ is of $\overset{(wt)}{\text{weight}}$ $m \in \mathbb{Z}$
if $d A_k \subset A_{k+m}$ $(A_{<0} := 0)$.

Ex 1: 1) $W : A \to A$ "weight operator"

$A_k \ni f \mapsto kf$

is derivation of wt 0.

2) $A = \widetilde{M} := \mathbb{Q}[G_2, G_4, G_6] \subset \mathbb{Q}[[q]]$

quasimodular forms

$$G_k = -\frac{B_k}{2k!} + \frac{1}{(k-1)!} \sum_{n>0} \frac{n^{k-1} q^n}{1-q^n}$$

Fact: $d = q\frac{d}{dq}$ is derivation of wt 2.

e.g. $q\frac{d}{dq} G_2 = 5 G_4 - 2 G_2^2$

Negative wt derivations $\rightsquigarrow$ polynomial repr.

<u>Lemma 1</u> Let $d \in \mathrm{Der}(A)$ be of wt $m \in \mathbb{Z}_{<0}$ and $a \in A$ with $d(a) = 1$.

Then $\forall f \in A$ $\exists!\, f_1, \dots, f_n \in \mathrm{Ker}(d)$ with

$$f = \sum_{i=0}^{n} f_i \odot a^{\odot i} \quad (\star)$$

(Converse also true: If $\forall f \in A$ $\exists!\, f_1, \dots, f_n \in A$ s.th. $(\star)$ then

$$d(f) := \sum_{i=0}^{n} i\, f_i \odot a^{\odot i-1}$$

is derivation with $d(a) = 1$.)

<u>Ex 2</u> $A = \mathfrak{h}^1 \quad \mathbb{Q}\langle z_k \mid k \geq 1 \rangle$

$\odot = *$ (or $ш$)

$$d(z_{k_1} \cdots z_{k_r}) = \begin{cases} z_{k_2} \cdots z_{k_r} & k_1 = 1 \\ 0 & \text{else} \end{cases}$$

Check: $d \in \mathrm{Der}(\mathfrak{h}^1_*)$

$\mathrm{Ker}(d) = \mathrm{span}\{z_{k_1} \cdots z_{k_r} \mid k_1 \geq 2,\ r \geq 0\} = \mathfrak{h}^0$

Lemma 1 $\Rightarrow$ $\mathfrak{h}^1 = \mathfrak{h}^0[z_1] \rightsquigarrow$ MZV reg.

(cf. Wang-san $A^1_* \ (N=1) = A^0_*[Y_1]$)

A is <u>$\mathfrak{sl}_2$-algebra</u> $:\Longleftrightarrow$ $\exists$ Lie alg. hom.

$$\mathfrak{sl}_2 \longrightarrow \mathrm{Der}(A)$$

$\Longleftrightarrow$ $\exists$ $\delta, W, D \in \mathrm{der}(A)$ s.th.

$$[W, D] = 2D, \quad [W, \delta] = -2\delta, \quad [\delta, D] = W.$$

$$(\delta, W, D) = \text{"}\mathfrak{sl}_2\text{-triple"}$$

<u>Ex 3</u> $\quad \widetilde{M} = \mathbb{Q}[G_2, G_4, G_6]$ is $\mathfrak{sl}_2$-alg.

wt	
-2	$\delta(G_2) = -\frac{1}{2}, \quad \delta(G_4) = \delta(G_6) = 0$
0	$W(G_k) = k\,G_k$
2	$D = q\frac{d}{dq}$

$$\mathrm{Ker}\,\delta = \mathbb{Q}[G_4, G_6] =: M \quad \text{Modular forms}$$

$\mathfrak{sl}_2$ structure $\rightsquigarrow$ Rankin-Cohen brackets $\overset{f \cdot g \text{ if } n=0}{[f,g]_n} \in M_{\ell+k+n}$

$f \in M_k, \; g \in M_\ell \qquad n \geq 0$

② Quasi-shuffle algebras

$\mathcal{L}$: set "letters"

$\Bbbk\mathcal{L}$: space of letters

$\diamond : \Bbbk\mathcal{L} \times \Bbbk\mathcal{L} \to \Bbbk\mathcal{L}$ com. & ass. product

$\rightsquigarrow (\Bbbk\mathcal{L}, \diamond)$ (non-unital) $\Bbbk$-alg.
"algebra of letters"

$\Bbbk\langle\mathcal{L}\rangle$: "space of $\underset{\text{monic monomial}}{\overset{!!}{\text{words}}}$"

$\Bbbk\langle\mathcal{L}\rangle$ ↙ ↘ Today

$(\Bbbk\langle\mathcal{L}\rangle, \cdot)$ — $\cdot$: concatination

non-com alg.

alg. gen: $\mathcal{L}$

$(\Bbbk\langle\mathcal{L}\rangle, *_\diamond)$ — $*_\diamond$: quasi-shuffle

com. alg.

alg. gen: Lyndon words in $\mathcal{L}$

empty word

Def. of $*_\diamond$: $1 *_\diamond w = w *_\diamond 1 = w$

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a \diamond b)(w *_\diamond v)$$

$a, b \in \mathcal{L}$, w, v words (cf. Wang-sans talk)

(Hoffman-Ihara : QSPs revisited)

special case: $a \diamond b = 0 \rightsquigarrow *_\diamond = ш$

<u>Ex 4</u> 1) $\mathcal{L}_{xy} = \{x, y\}$, $\mathfrak{h} = \mathbb{Q}\langle \mathcal{L}_{xy} \rangle$

$$\cup$$

$$\mathfrak{h}' = \mathbb{Q} + \mathfrak{h} y$$

$$\cup$$

$$\mathfrak{h}^0 = \mathbb{Q} + x \mathfrak{h} y$$

$$z_k \mapsto \underbrace{x \cdots x}_{k-1} y$$

2) $\mathcal{L}_z = \{ z_k \mid k \geq 1 \}$ $\quad \mathbb{Q}\langle \mathcal{L}_z \rangle \cong \mathfrak{h}'$

$$z_{k_1} \diamond z_{k_2} = z_{k_1 + k_2} \rightsquigarrow *_\diamond = *$$

(MZV: $\bullet \in \{*, ш\}$

$$(\mathfrak{h}^0, \bullet) \longrightarrow \mathbb{R} \quad \text{alg. hom}$$

$$z_{k_1} \cdots z_{k_r} \longmapsto \zeta(k_1, \ldots, k_r)$$
)

If time enough

3) $\hat{\mathcal{L}}_z = \mathcal{L}_z \cup \{z_0\}$

$$z_{k_1} \triangle z_{k_2} = \begin{cases} z_{k_1+k_2} & k_1, k_2 \geq 1 \\ 0 & \text{else } (k_1 \cdot k_2 = 0) \end{cases}$$

$$(\mathbb{Q}\langle \hat{\mathcal{L}}_z \rangle, *_0)$$

$\cup$ sub. alg.

$$\hat{\mathfrak{h}}' := \mathrm{span}\{ z_{k_1} \cdots z_{k_r} \mid k_1 \geq 1 \}$$

③ Derivations for QSA

Motivation: Understand $\mathrm{Der}((\mathbb{K}\langle\mathcal{L}\rangle, *_0))$

For $\varphi: \mathbb{K}\mathcal{L} \to \mathbb{K}\mathcal{L}$ lin.

define $\Theta^{\varphi}: \mathbb{K}\langle\mathcal{L}\rangle \to \mathbb{K}\langle\mathcal{L}\rangle$

$$a_1 \cdots a_r \longmapsto \sum_{j=1}^{r} a_1 \cdots \varphi(a_j) \cdots a_r .$$

Prop 2 ([BIM]) The map

$$\mathrm{Der}((\mathbb{K}\mathcal{L}, \diamond)) \longrightarrow \mathrm{Der}((\mathbb{K}\langle\mathcal{L}\rangle, *_0))$$
$$\varphi \longmapsto \Theta^{\varphi}$$

is a Lie algebra homomorphism.

Ex 5: $\varphi: \mathbb{Q}\mathcal{L}_z \to \mathbb{Q}\mathcal{L}_z$ (clearly not surjective)

$$z_k \longmapsto k\, z_{k+1}$$

$$\varphi(z_{k_1} \diamond z_{k_2}) = \varphi(z_{k_1}) \diamond z_{k_2} + z_{k_1} \diamond \varphi(z_{k_2})$$

$$\varphi(z_{k_1+k_2}) = (k_1+k_2)\, z_{k_1+k_2+1}$$

Prop2 $\Rightarrow$ $\Theta^{\ell}: z_{k_1}\cdots z_{k_r} \longmapsto \sum_{j=1}^{r} k_j \, z_{k_1}\cdots z_{k_j+1}\cdots z_{k_r}$

is derivation on $(\mathfrak{h}^1, *)$

In [BIM] we give several other families of derivations on $(\mathbb{Q}\langle\mathcal{L}\rangle, *_0)$. (Formulas are a bit messy.)

Here are some easy examples:

Prop 3 ([BIM]) i) For any $a \in \mathcal{L}$ the lin. map

$$\Theta_0^{[a}: \mathbb{Q}\langle\mathcal{L}\rangle \longrightarrow \mathbb{Q}\langle\mathcal{L}\rangle$$

$$a_1\cdots a_r \longmapsto \sum_{1\leq \ell\leq r} \mathbb{1}_{a_1 \diamond \cdots \diamond a_\ell = a} \frac{(-1)^{\ell+1}}{\ell} a_{\ell+1}\cdots a_r$$

is a derivation on $(\mathbb{Q}\langle\mathcal{L}\rangle, *_0)$

(ex: $a = z_1$, $*$ $\rightsquigarrow$ Ex2)

ii) $a \in \mathcal{L}$, $\varphi : \mathbb{Q}\mathcal{L} \to \mathbb{Q}\mathcal{L}$ lin. map.

$$\Theta^{\varphi, a} : \mathbb{Q}\langle\mathcal{L}\rangle \longrightarrow \mathbb{Q}\langle\mathcal{L}\rangle$$

$$a_1 \cdots a_r \longmapsto \sum_{j=1}^{r-1} \mathbb{1}_{a_j = a}\, a_1 \cdots a_{j-1}\, \varphi(a_{j+1}) \cdots a_r$$

$$- \sum_{j=2}^{r} \mathbb{1}_{a_j = a}\, a_1 \cdots \varphi(a_{j-1})\, a_{j+1} \cdots a_r$$

is derivation on $(\mathbb{Q}\langle\mathcal{L}\rangle, \underset{\underset{o \equiv o}{\uparrow}}{ш})$.

(There exist a version for $*_o$ of ii)).

④ Mot$\left[\begin{array}{l}\text{ivation}\\ \text{iple Eisenstein series}\end{array}\right.$

	Numbers		Functions
Single	$\zeta(k)$	$\xleftarrow{q=0}$	$\mathbb{G}_k(\tau) = \zeta(k) + \sum_{n>0} a_n q^n$ $\quad \mathfrak{sl}_2$
Multiple	$\zeta(k_1,\dots,k_r)$	$\xleftarrow{q=0}$	$\mathbb{G}_{k_1,\dots,k_r}(\tau)$ $\quad ?\ \mathfrak{sl}_2$
	$\Vert ? ?$		$\Vert ? ?$ $\quad \mathfrak{sl}_2$ ✓ [BIM]
Formal	$\zeta^f(k_1,\dots,k_r)$	$\xleftarrow{\pi^f}$	$G^f(k_1,\dots,k_r)$

Formal MZV $\qquad Z^f := (\mathfrak{b}', *) \,/\, \text{EDS}$

FMES $\qquad \mathcal{G}^f := (\mathfrak{b}', *) \,/\, ?$

Consider $\hat{\mathcal{L}}_Z = \mathcal{L}_Z \cup \{z_0\}$

$$z_{k_1} \diamond z_{k_2} = \begin{cases} z_{k_1+k_2} & k_1, k_2 \geq 1 \\ 0 & \text{else } (k_1 \cdot k_2 = 0) \end{cases}$$

$$(\mathbb{Q}\langle \hat{\mathcal{L}}_Z \rangle, *_\diamond)$$

$\cup$ sub. alg.

$$\mathfrak{h}^1_* \subset \hat{\mathfrak{h}}^1 := \operatorname{span}\{ z_{k_1} \cdots z_{k_r} \mid k_1 \geq 1 \}$$

Define the lin. map. $\sigma : \hat{\mathfrak{h}}^1 \longrightarrow \hat{\mathfrak{h}}^1$ by

$k_1, \ldots, k_r \geq 1$, $m_1, \ldots, m_r \geq 1$

$$\sigma(z_{k_1} z_0^{m_1 - 1} \cdots z_{k_r} z_0^{m_r - 1}) = z_{m_r} z_0^{k_r - 1} \cdots z_{m_1} z_0^{k_1 - 1}$$

I : $*$-Ideal generated by $\sigma(w) - w$ $\forall w \in \hat{\mathfrak{h}}^1$.

FMES : $\hat{G}^f := (\hat{\mathfrak{h}}^1, *) / I$

$k_1, \ldots, k_r \geq 0$, $k_1 \geq 1$ $\quad G^f(k_1, \ldots, k_r) :=$ class of $z_{k_1} \cdots z_{k_r}$.

$\mathcal{G}^f = \operatorname{span}\{ G^f(k_1,\dots,k_r) \mid k_1,\dots,k_r \geq 1\} \subset \hat{\mathcal{G}}^f$

Conjecture: $\mathcal{G}^f = \hat{\mathcal{G}}^f$

σ- equivariant derivations on $(\hat{\mathfrak{h}}^1, *)$
$(d(w) = d(\sigma(w)))$

$\rightsquigarrow$ derivations on $\hat{\mathcal{G}}^f$

<u>Theorem</u> ([BIM])
i) $\mathcal{G}^f$ and $\hat{\mathcal{G}}^f$ are $\mathfrak{sl}_2$- algebras
ii) $\widetilde{\mathcal{M}}^f := \mathbb{Q}[G^f(2), G^f(4), G^f(6)] \underset{\substack{\uparrow \\ \text{as } \mathfrak{sl}_2\text{-alg.}}}{\cong} \widetilde{M}$

On $\mathcal{G}^f$: $D(G^f(w)) = G^f(w * z_2 - w * z_2)$

$$\begin{aligned}
\delta\, G^f(k_1,\dots,k_r) = &-\frac{1}{2}\mathbf{1}_{k_1=2}\, G^f(k_2,\dots,k_r) + \frac{1}{4}\mathbf{1}_{k_1=k_2=1} G^f(k_3,\dots,k_r) \\
&+\frac{1}{2}\sum_{j=1}^{r-1} \mathbf{1}_{\substack{k_j=1 \\ k_{j+1}>1}}\; G^f(k_1,\dots,k_{j-1},k_{j+1}-1,\dots,k_r) \\
&-\frac{1}{2}\sum_{j=2}^{r} \mathbf{1}_{\substack{k_j=1 \\ k_{j-1}>1}}\; G^f(k_1,\dots,k_{j-1}-1,k_{j+1},\dots,k_r).
\end{aligned}$$