$1\quad 2\quad 3\quad \pi\quad e\quad \sum_{n=1}^{\infty}\frac{1}{n^2}$

Numbers, infinite sums and multiple zeta values

12th JSPS Club Meeting in Japan / December 16, 2023

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Who am I

Born & Studied in Hamburg (-2015)

2016: JSPS Postdoctoral Fellow in Nagoya

JSPS Science Dialogue, Oshu (Iwate)

2017: MPIM Bonn

2018: YLC Assistant Prof., Nagoya

2019: G30 Program Designated Associate Prof., Nagoya

Since 2022: Associate Prof., Graduate School of Mathematics, Nagoya

Always interested in connecting Japan to Germany/Europe

JAPAN EUROPE
NUMBER THEORY EXCHANGE SEMINAR

nu Nagoya University
Mirai Education Project
名大みらい育成プロジェクト

NU-EMI

ChatGPTs interpretation of “a german number theorist working and living in Japan”

Goal of this talk

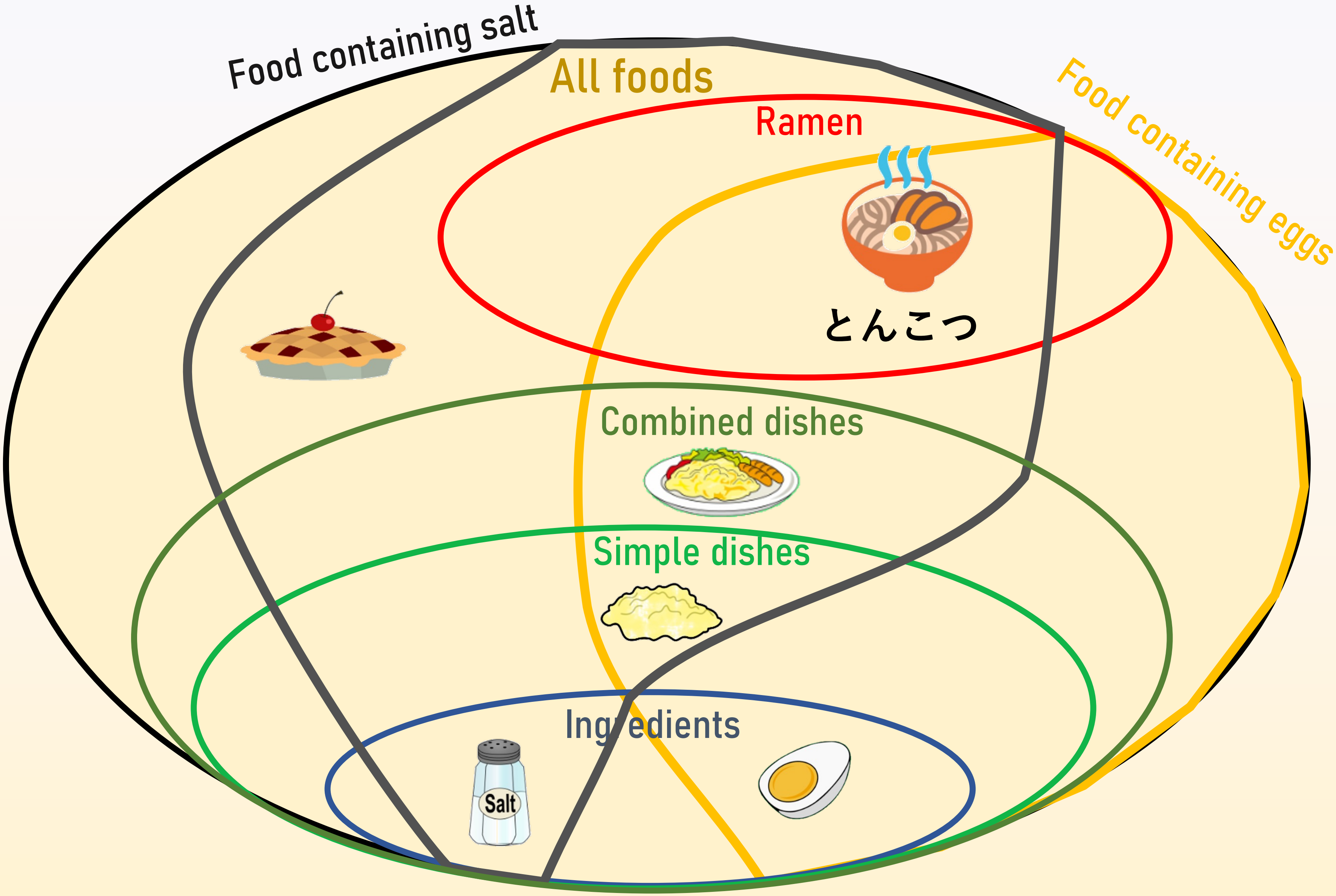
1. Talk about the classification of numbers

   What does that mean?

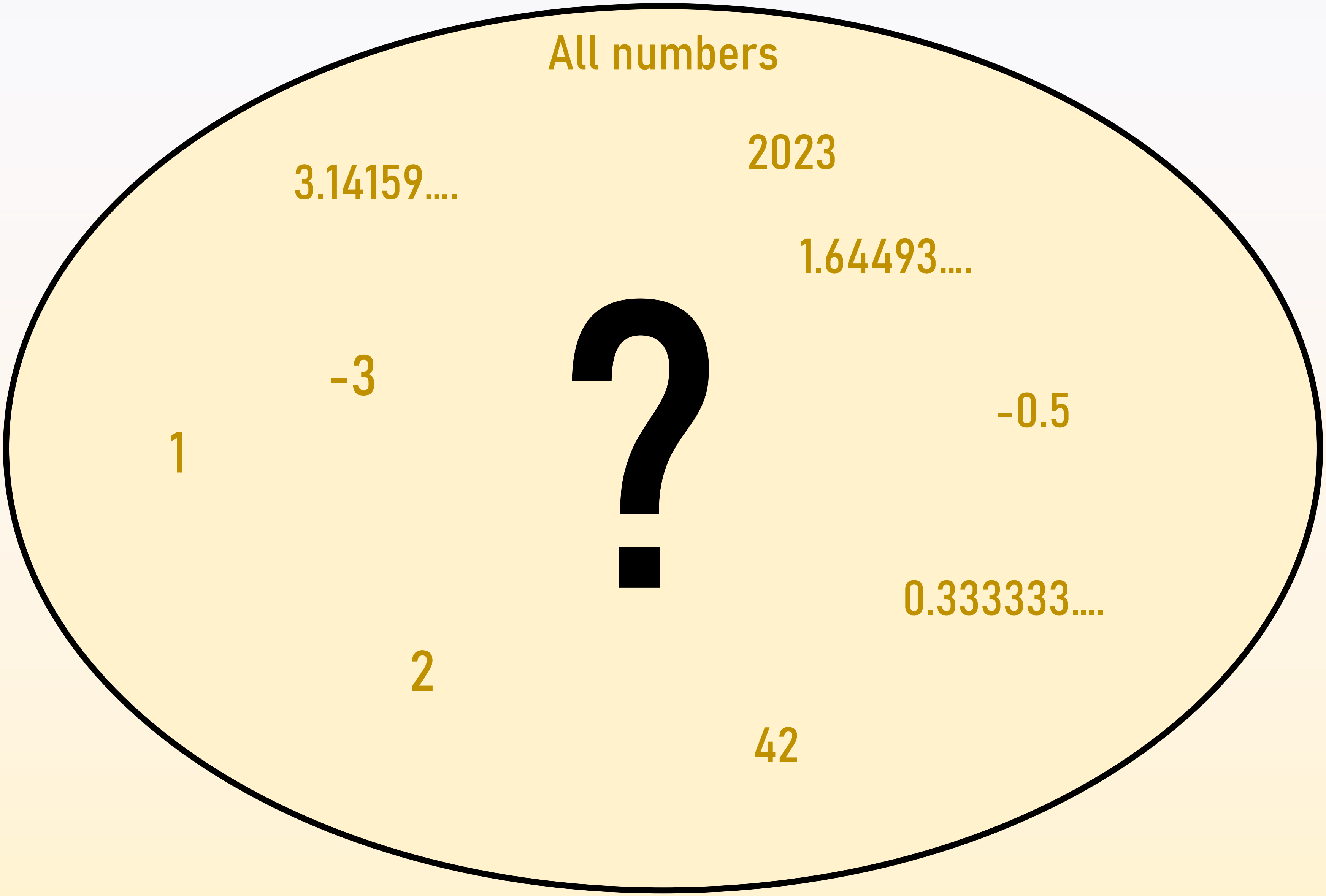
2. Infinite sums

3. Multiple zeta values

Classification of food



Classification of numbers

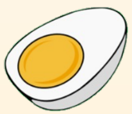


The numbers 1,2,3,4,.... are called natural numbers

We can add (+) and multiply (∗) natural numbers

$$6 = 2 * 3$$

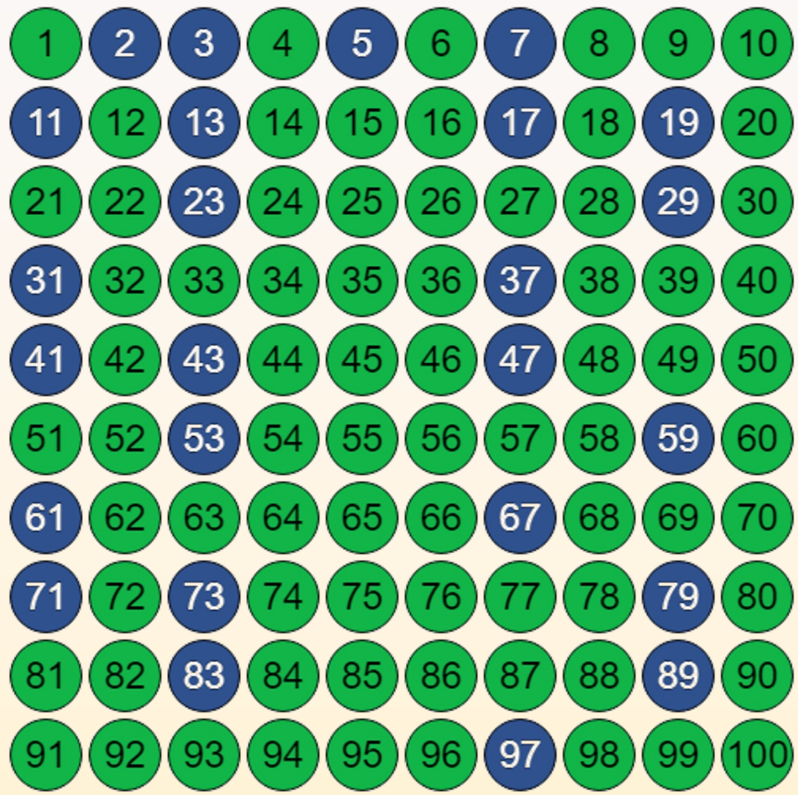
 = 

∗ 

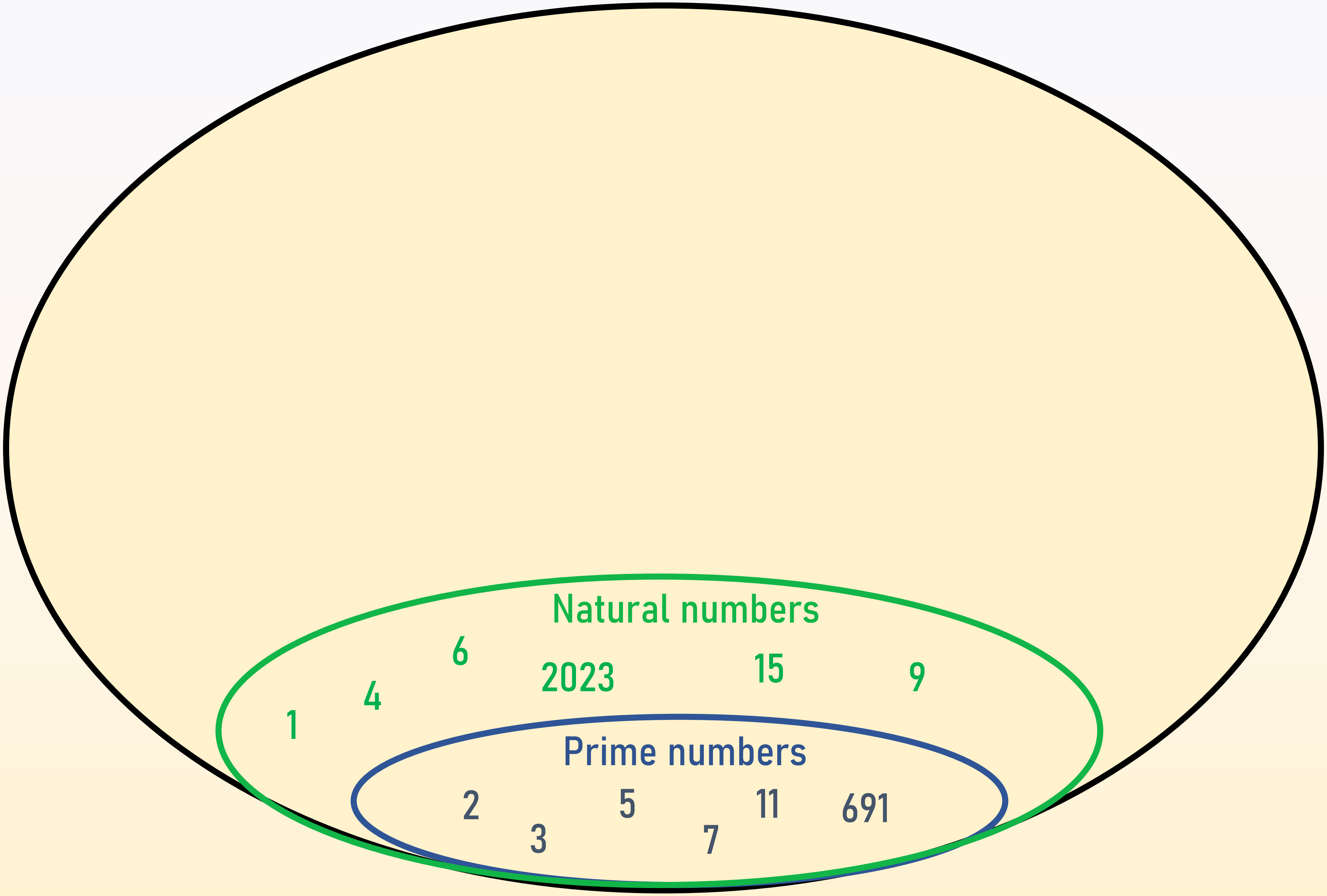
What are the ingredients for natural numbers?

Prime numbers the ingredients...

A natural number greater than 1 is called a **prime number**, if it can not be written as a product of two smaller numbers.



Classification of numbers ... so far



Integers

We also have zero 0 and negative numbers -1, -2, -3, -4,...

The natural numbers together with 0 and their negatives are called **integers**.

Mathematicians point of view

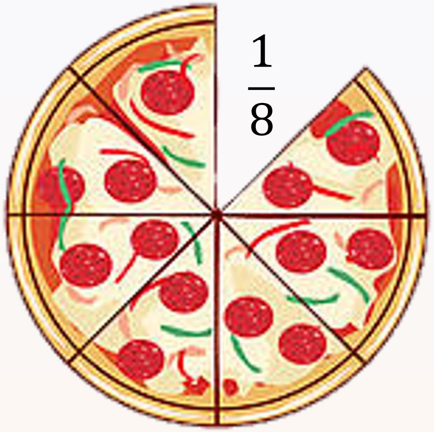
A, B : natural numbers

Integers allow us to solve the following equation for X:

$$X + B = A$$

For example $X = -5$ is the solution of $X + 7 = 2$.

Rational numbers combined dishes...



$\frac{1}{8}$

$\frac{7}{8}$

Numbers given by fractions are called **rational numbers**.

$$\frac{a}{b}$$

a ← numerator

b ← denominator

<u>Mathematicians point of view</u>

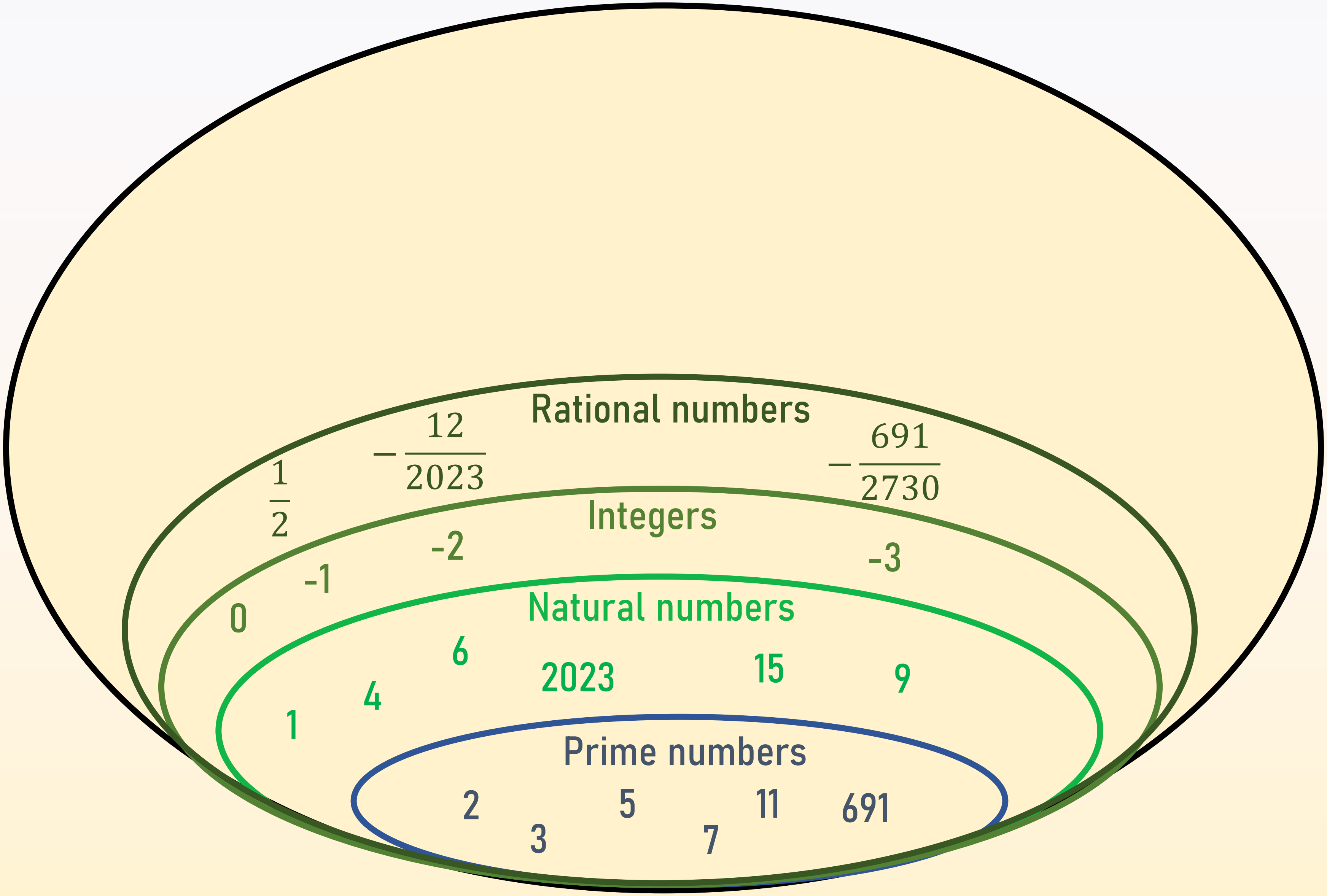
A, B, C : natural numbers

Rational numbers allow us to solve the following equation for X:

$$C\,X + B = A$$

For example $X = \frac{2}{5}$ is the solution of $5\,X + 7 = 9$.

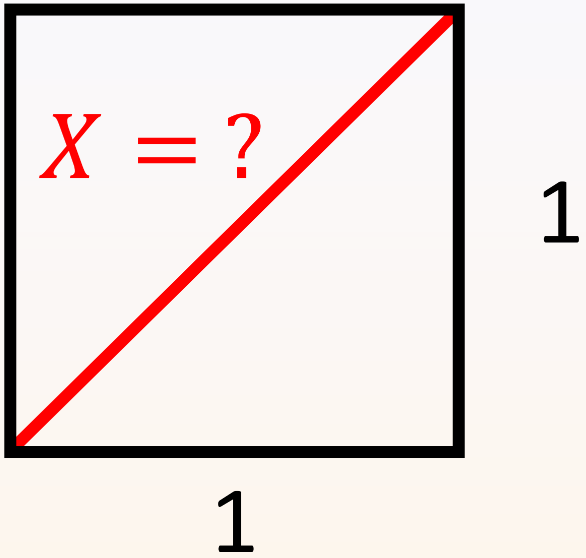
Classification of numbers ... so far



Algebraic numbers

$$X^2 = 1^2 + 1^2 = 2$$

$$X = \sqrt{2} \approx 1.414\ldots$$



Mathematicians point of view

$A_0, A_1, \ldots, A_n$: integers

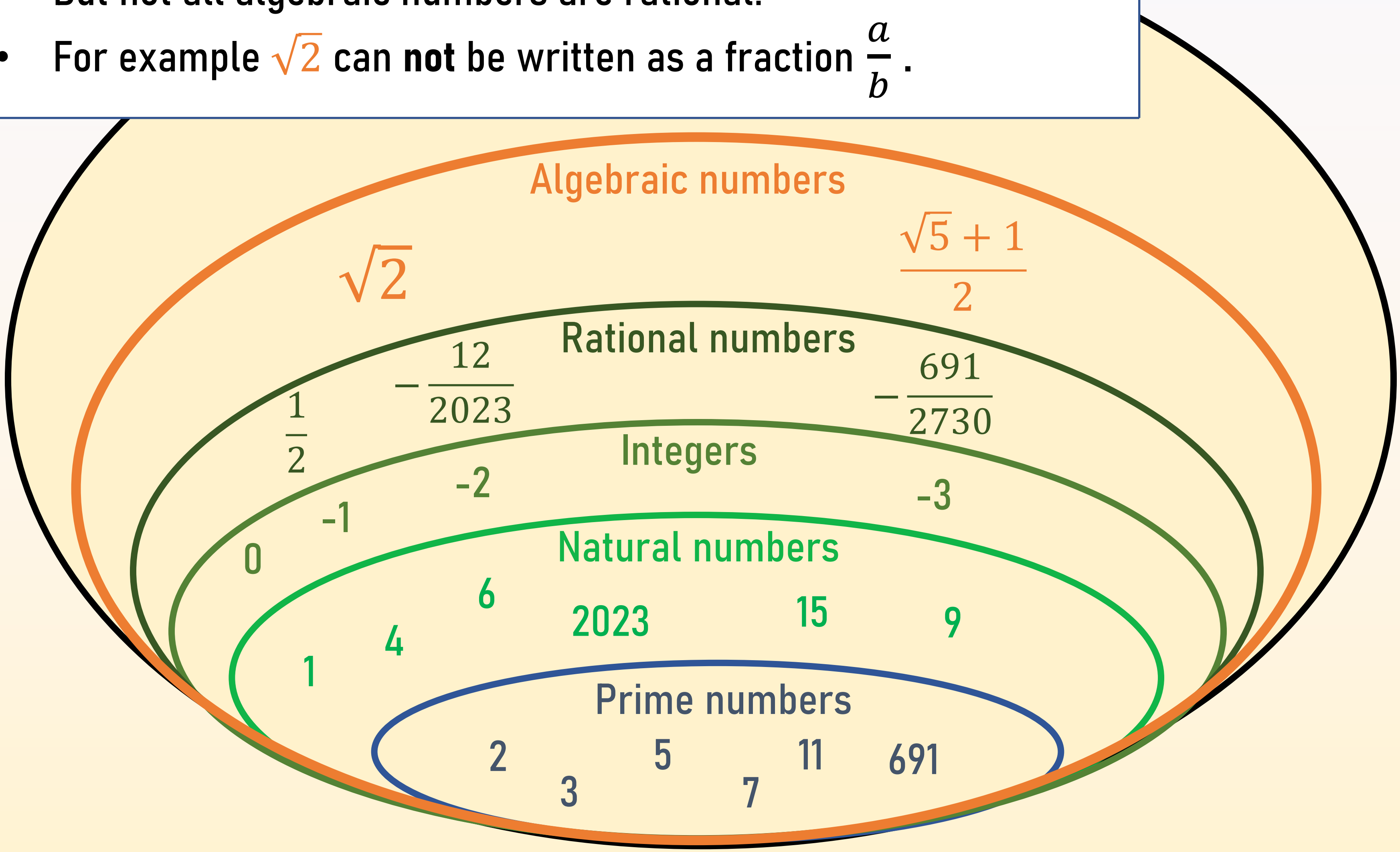
Algebraic numbers are given as solutions for X of polynomial equations:

$$A_n X^n + \ldots + A_2 X^2 + A_1 X + A_0 = 0$$

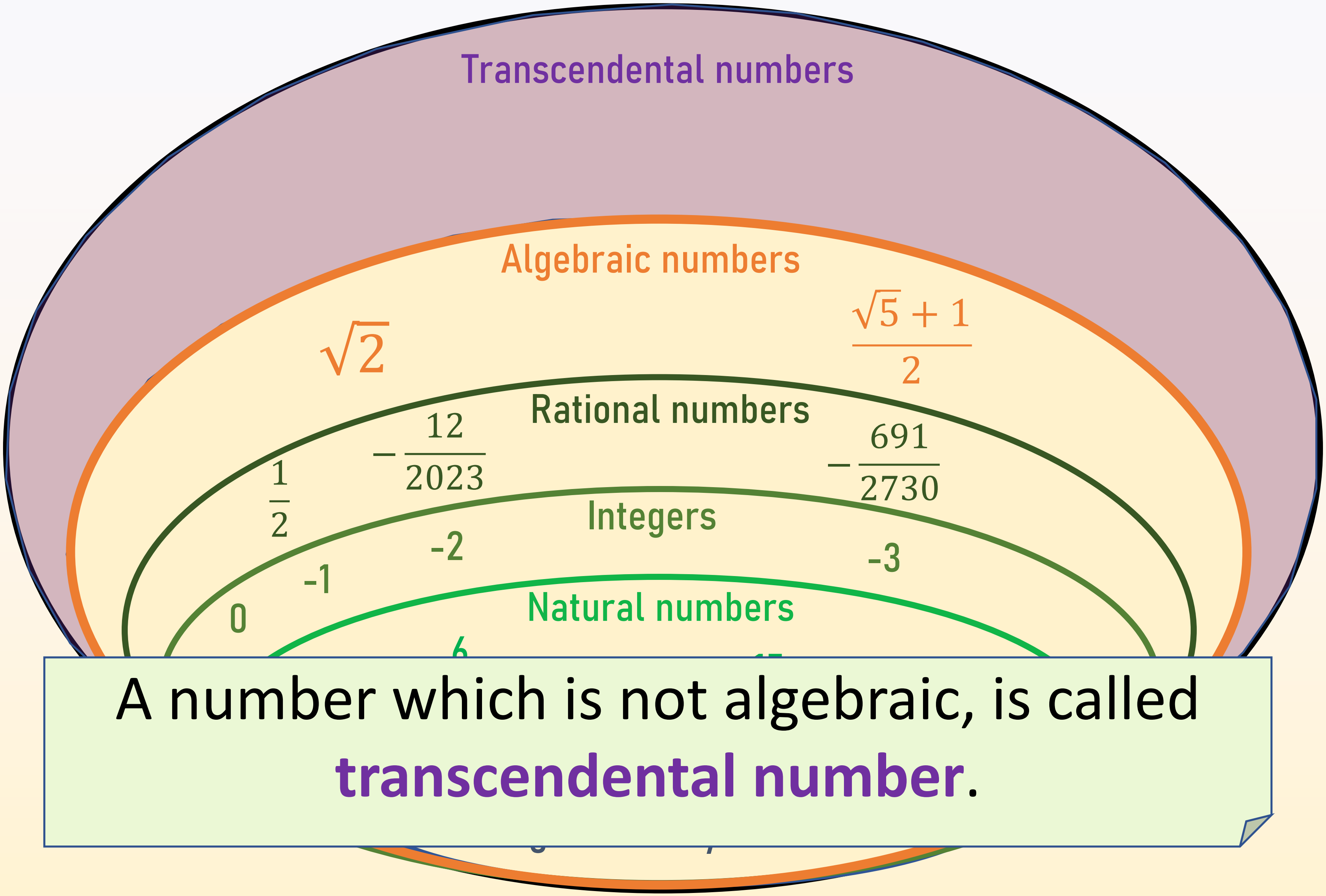
For example $X = \sqrt{2}$ is the solution of $1\,X^2 + 0\,X - 2 = 0$.

Classification of numbers ... so far

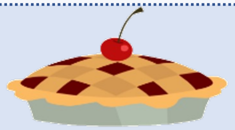
- Every rational number is also an algebraic number.
- But not all algebraic numbers are rational!
- For example $\sqrt{2}$ can **not** be written as a fraction $\frac{a}{b}$.

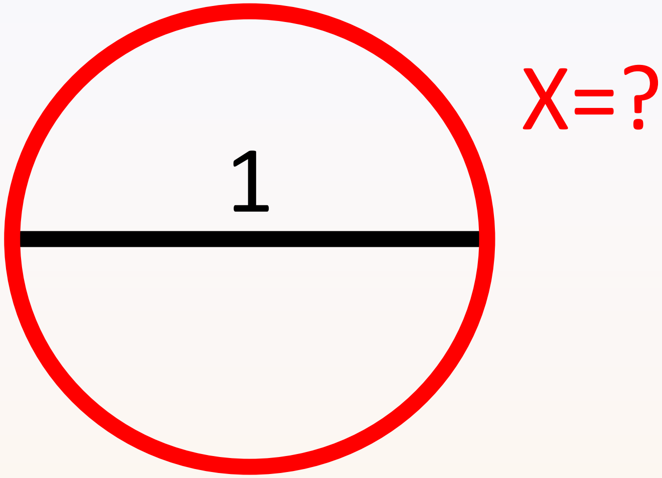


Transcendental numbers

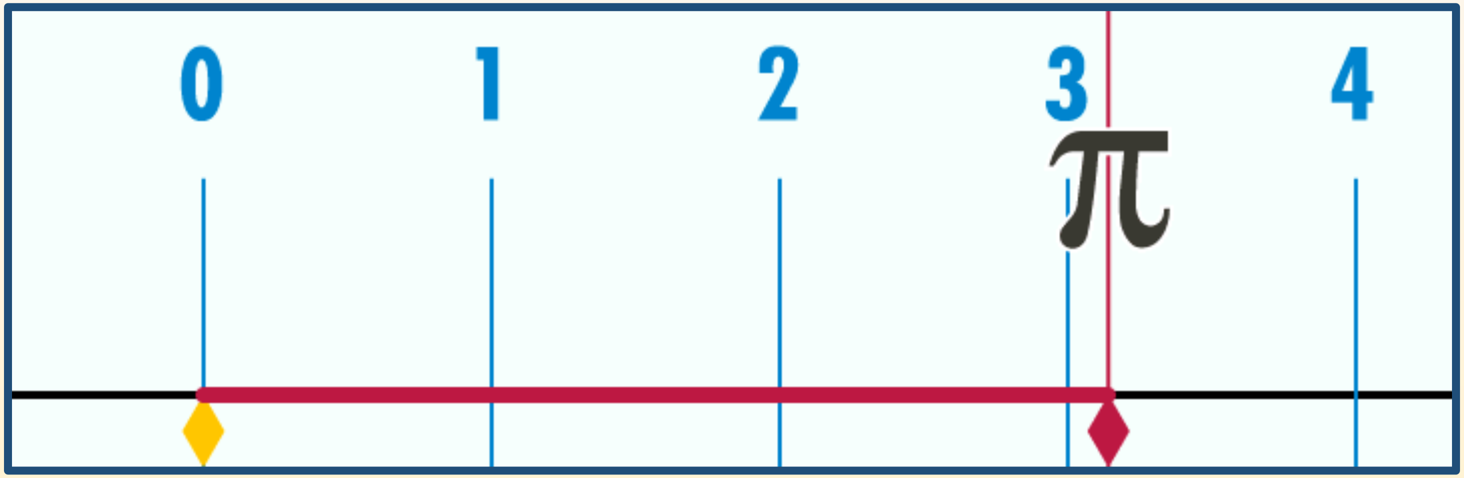


A number which is not algebraic, is called **transcendental number**.

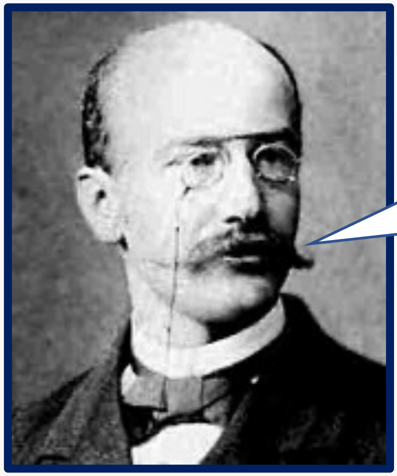




$$\pi = 3.141592.....$$



Pi π



Ferdinand von Lindemann
(1852 – 1939)

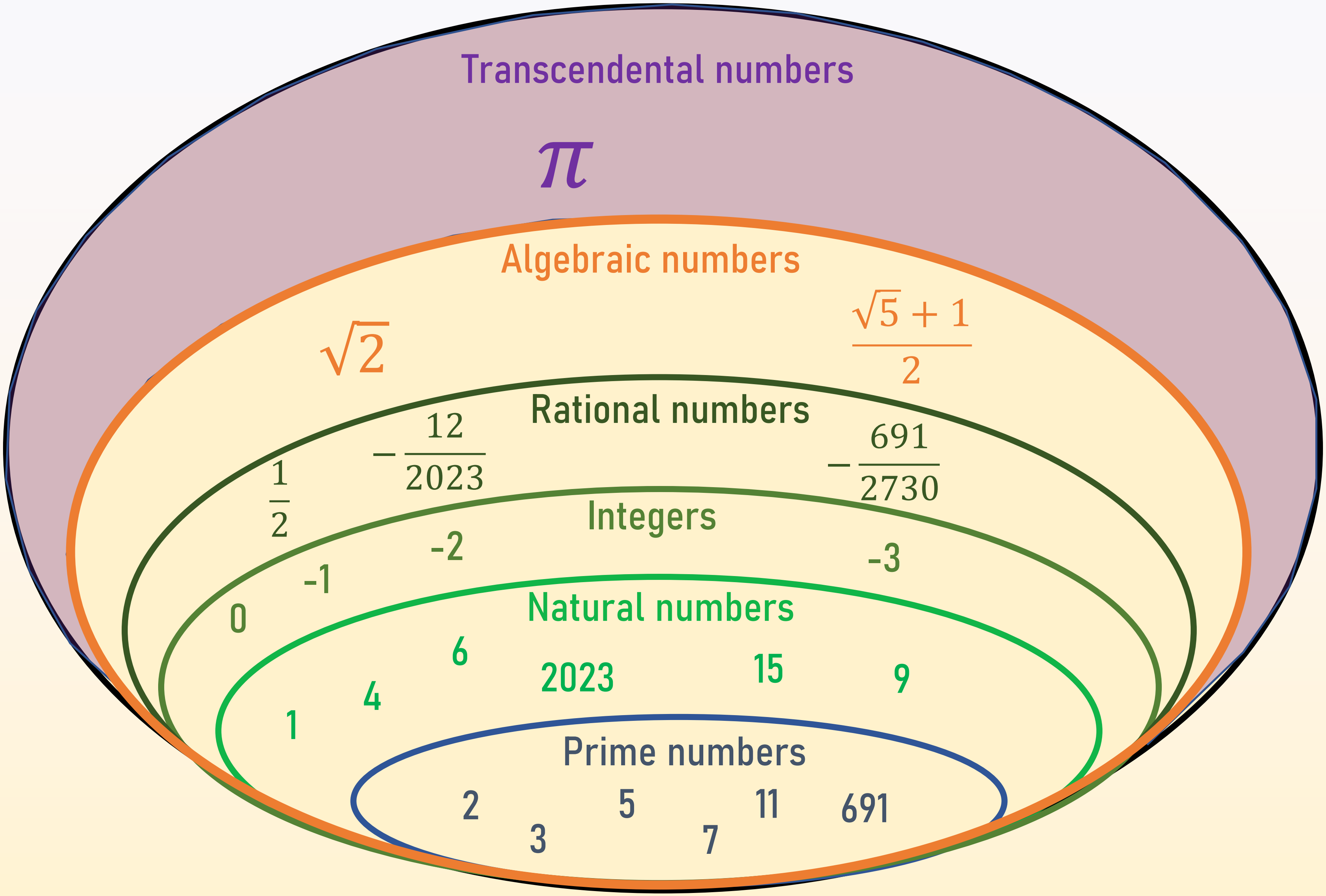
π is **transcendental**!

This means you will **never** find integers $A_0, A_1, \ldots, A_n$ such that

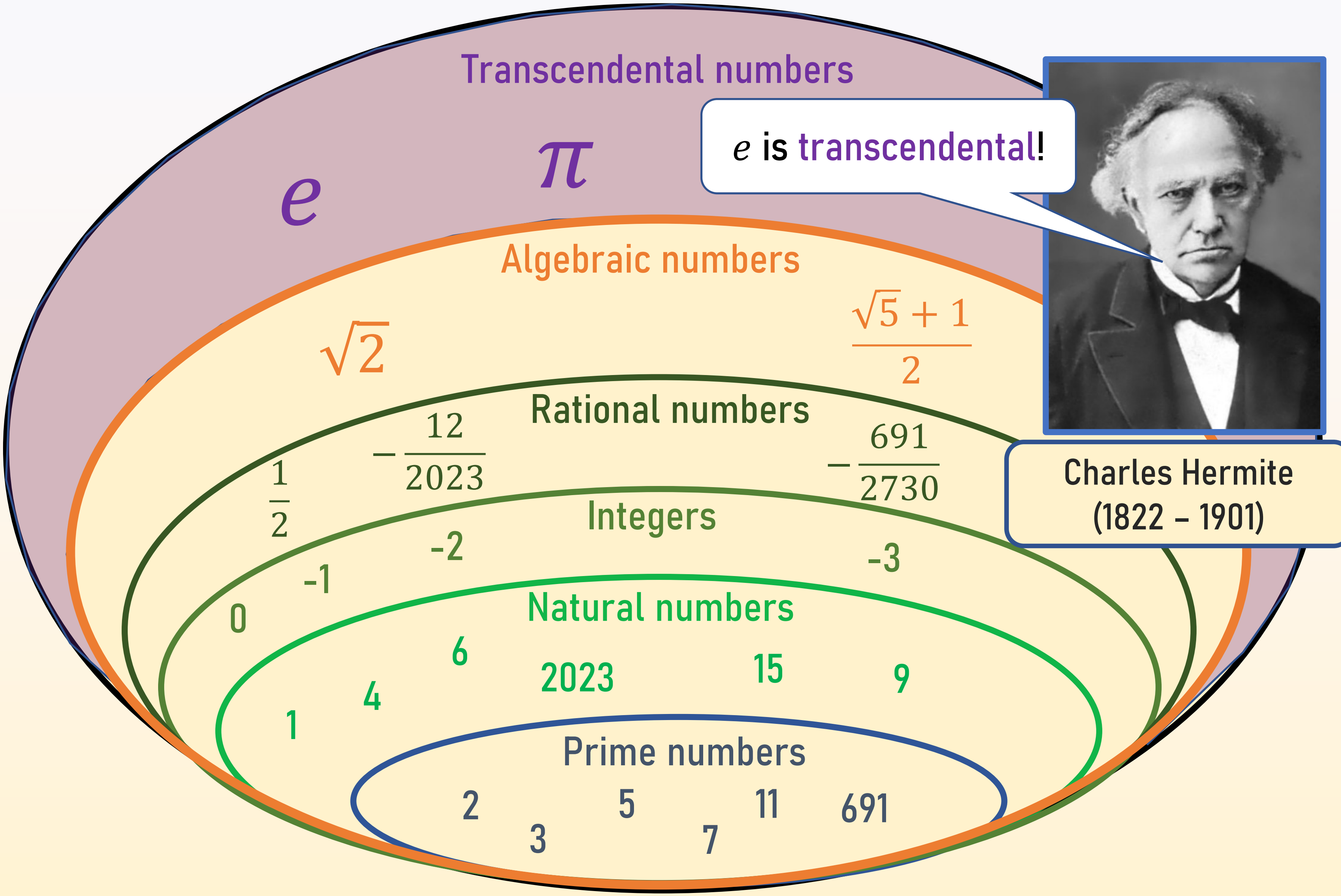
$$A_n \pi^n + \ldots + A_2\pi^2 + A_1\pi + A_0 = 0$$

$$7\pi - 22 \neq 0$$
$$2\pi^2 - 6\pi - 1 \neq 0$$
$$\pi^3 - 22\pi^2 + 4\pi - 12 \neq 0$$
....

Classification of numbers … so far



Classification of numbers ... so far



Infinite sums

Finite sums:

$1 = 1$

$1 + 2 = 3$

$1 + 2 + 3 = 6$

$1 + 2 + 3 + 4 = 10$

...

$1 + 2 + 3 + \ldots + 100 = 5050$

This sum gets bigger and bigger and therefore the infinite sum

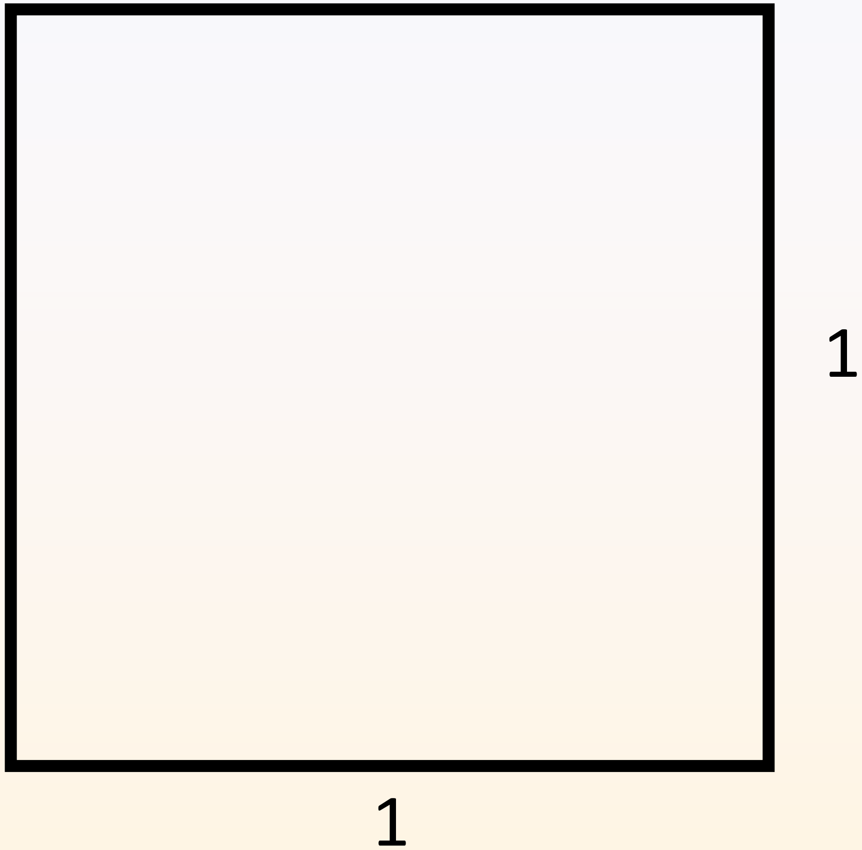
$$1 + 2 + 3 + 4 + \ldots + 100 + \ldots + 43432423 + \ldots\ldots$$

does not make sense.

But there are infinite sums which make sense!

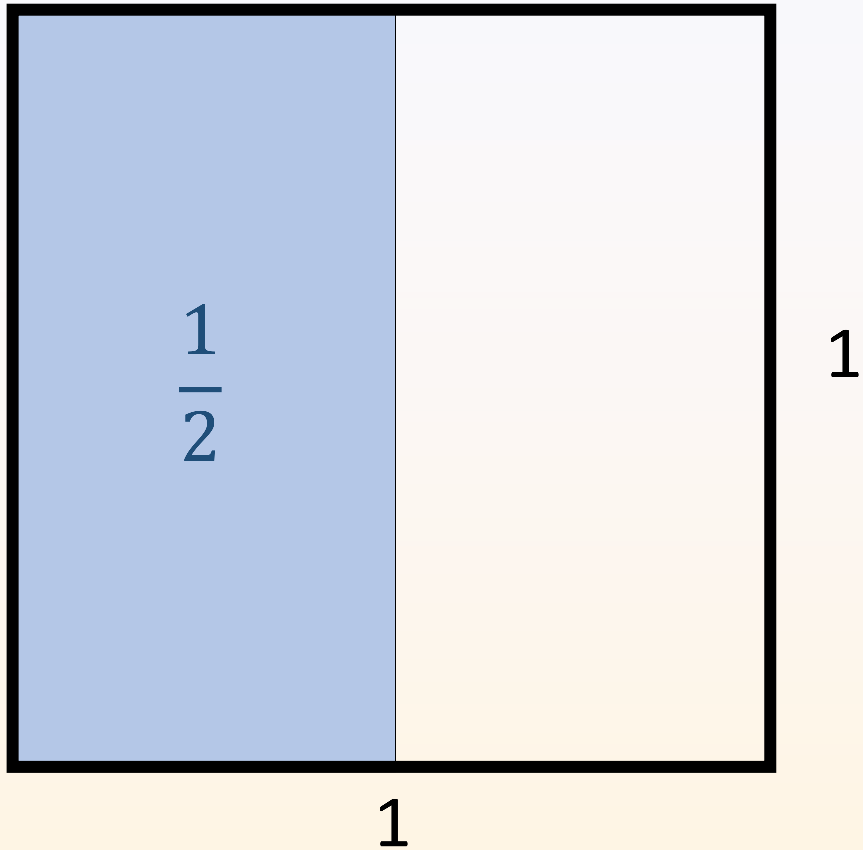
$$1 + 0.1 + 0.01 + 0.001 + 0.0001 + \ldots = 1.1111111\ldots = \frac{10}{9}$$

Infinite sums - Example



Area of □ = 1

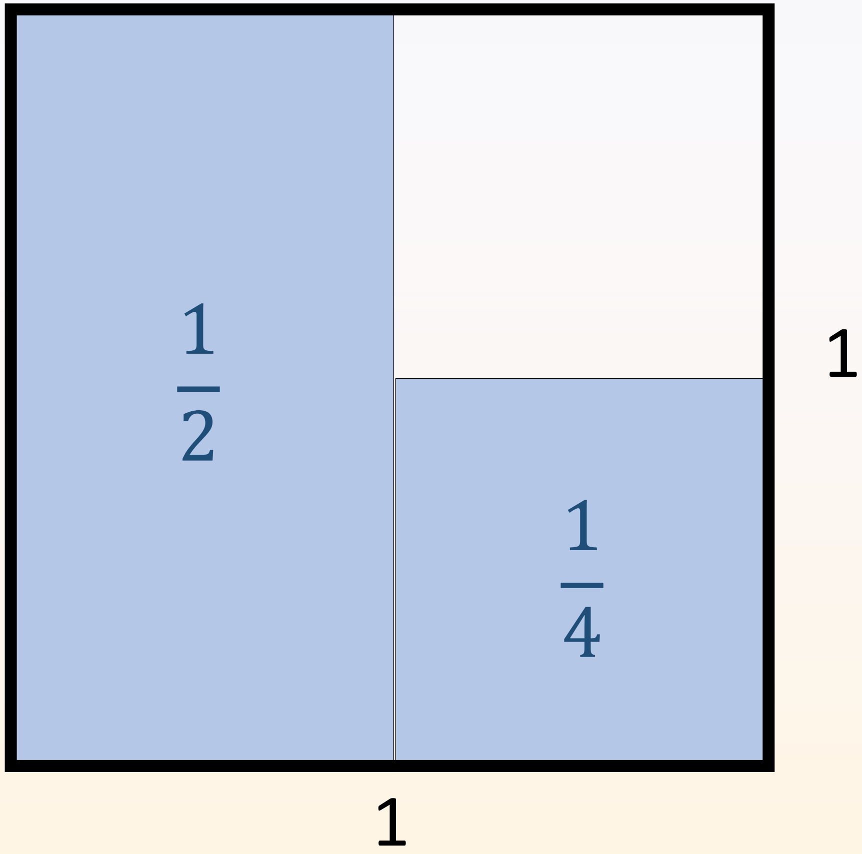
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2} = 0.5$

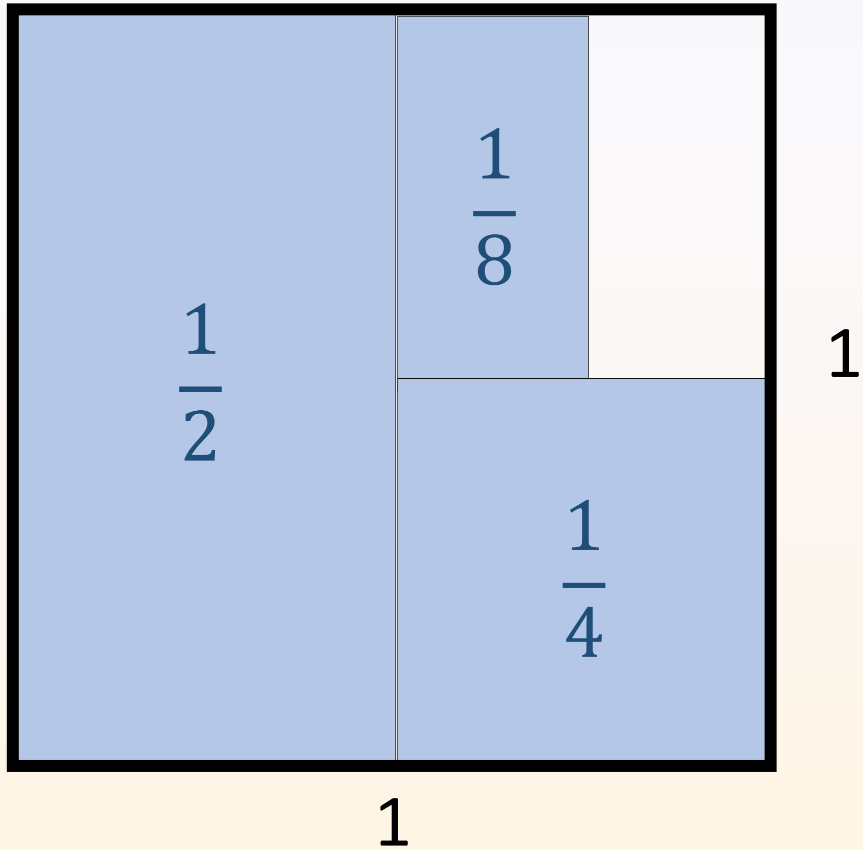
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2} + \frac{1}{4} = 0.75$

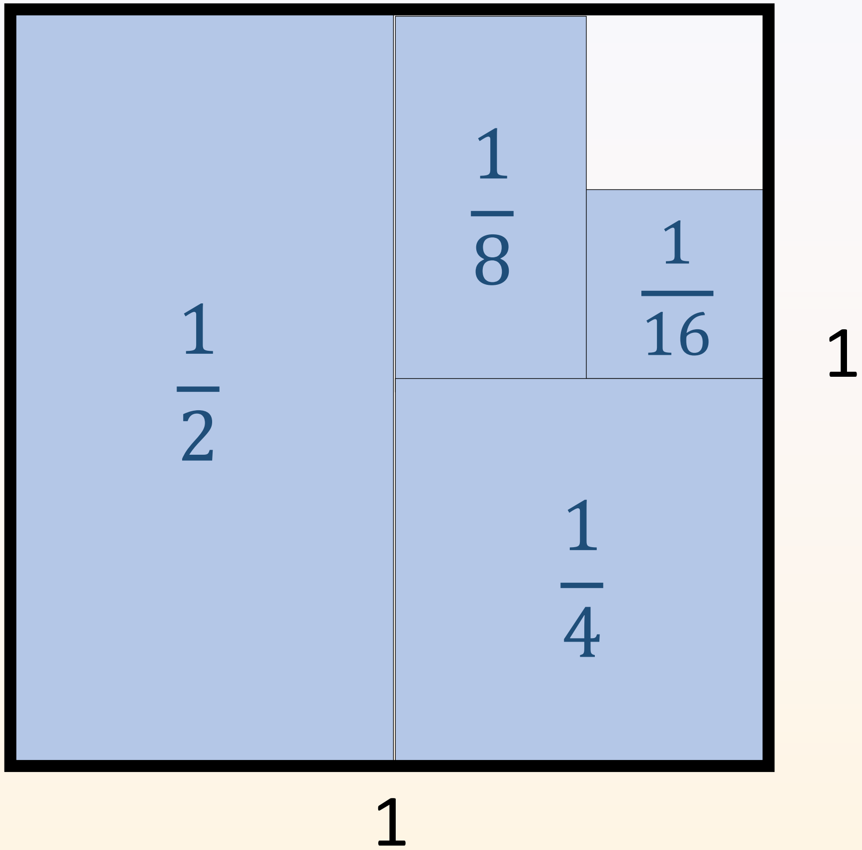
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} = 0.875$

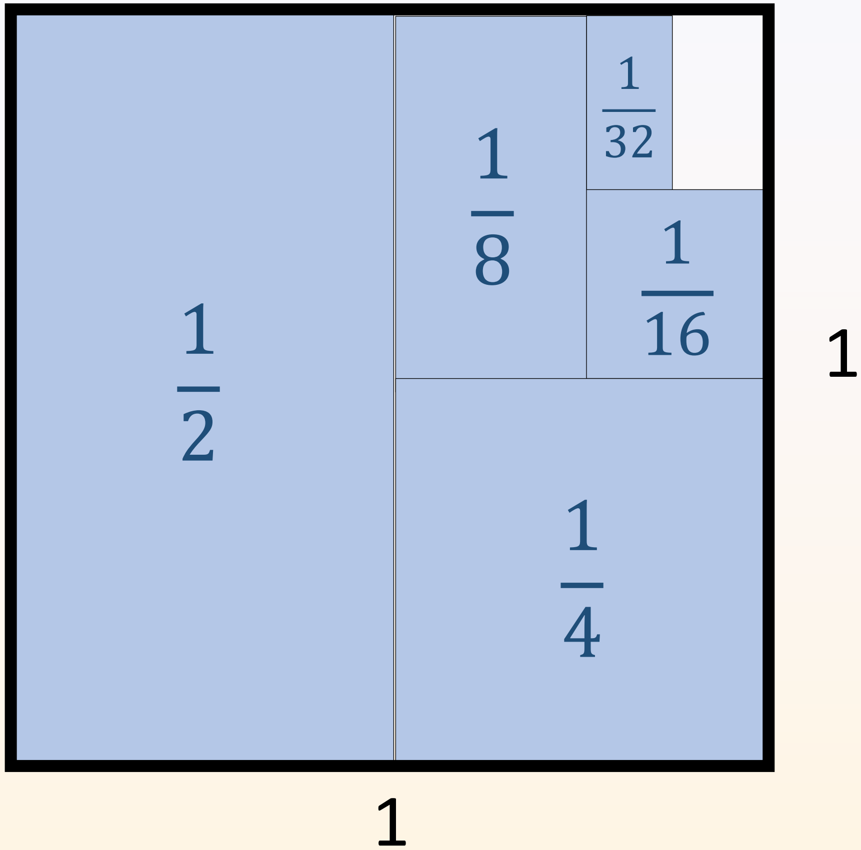
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = 0.93 \ldots$

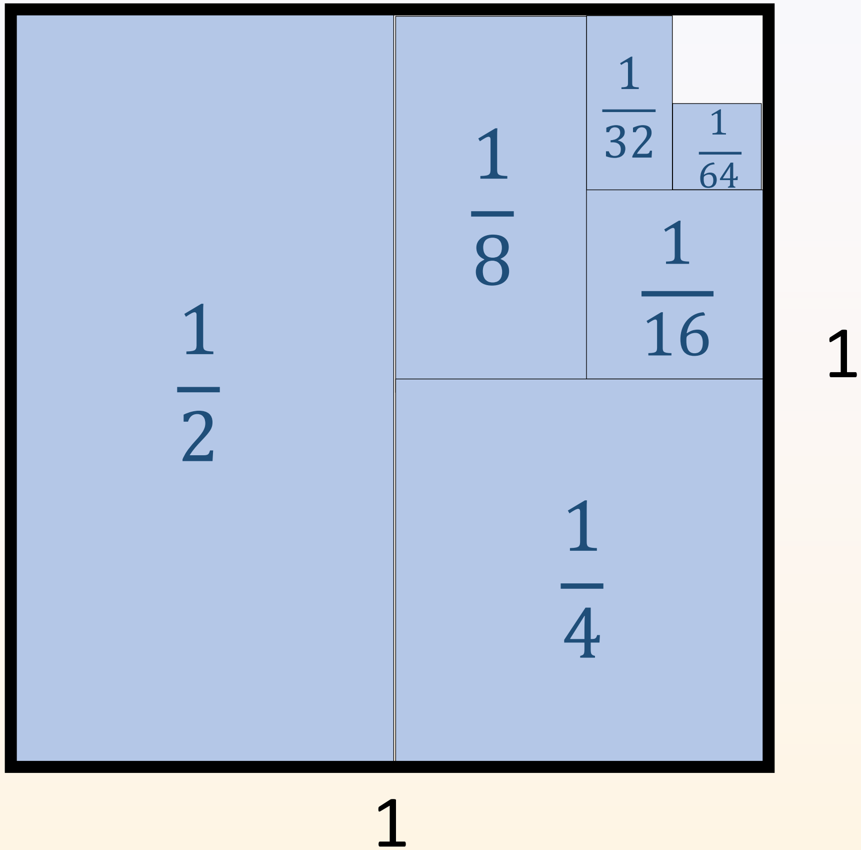
Infinite sums - Example



Area of □ = 1

Area of blue part $= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = 0.96\ldots$

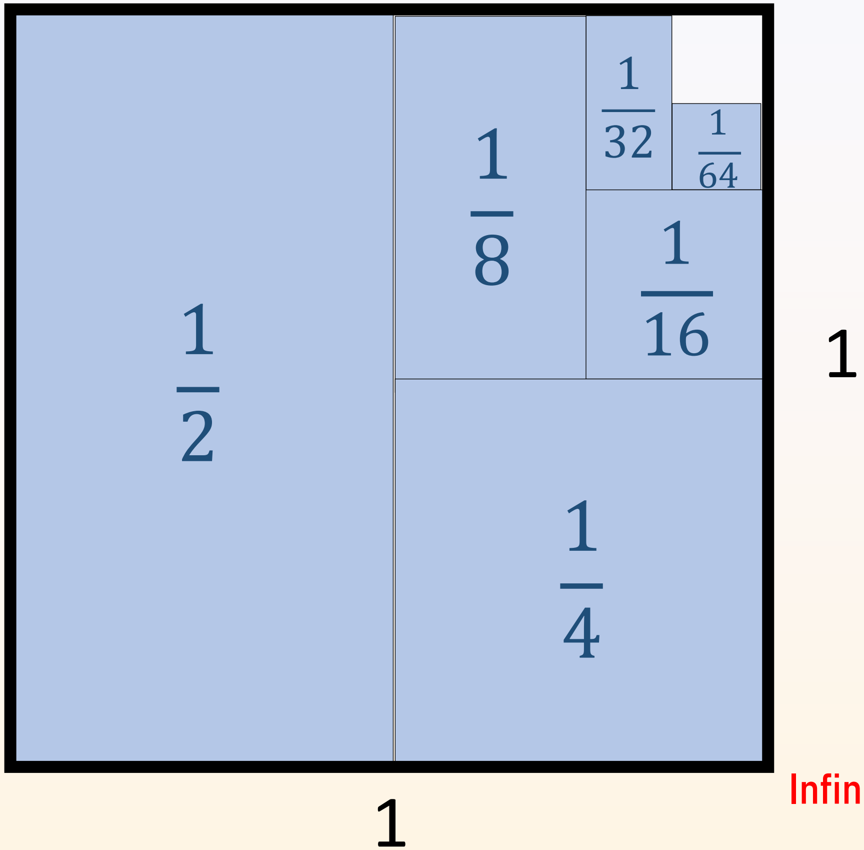
Infinite sums - Example



Area of □ = 1

Area of blue part = $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}=0.98\ldots$

Infinite sums - Example



Infinite number of terms (no end)

$$\text{Area of } \square = 1 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots$$

$$\text{Area of blue part} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} = 0.98\ldots$$

Geometric series

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\cdots=1$$

Mathematical notation:

$$\sum_{n=1}^{\infty}\frac{1}{2^n}=\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\cdots=1$$

For any natural number A greater than 1 we have

$$\sum_{n=1}^{\infty}\frac{1}{A^n}=\frac{1}{A-1}$$

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots = 1$$

What happens if we change this sum a little bit?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$= \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = ??$$

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = ???$$

$$\frac{1}{1^2} = 1$$

$$\frac{1}{1^2} + \frac{1}{2^2} = 1 + \frac{1}{4} = 1.25$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} = 1 + \frac{1}{4} + \frac{1}{9} = 1.3611\ldots$$

$$\frac{1}{1^2} + \cdots + \frac{1}{100^2} = 1 + \cdots + \frac{1}{10000} = 1.6349\ldots$$

$$???$$

Another infinite sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

Leonhard Euler
(1707 – 1783)

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^8} = \frac{\pi^8}{9450}$$

Riemann zeta values

For any natural number k greater than 1 the numbers

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots$$

are called **Riemann zeta values.**

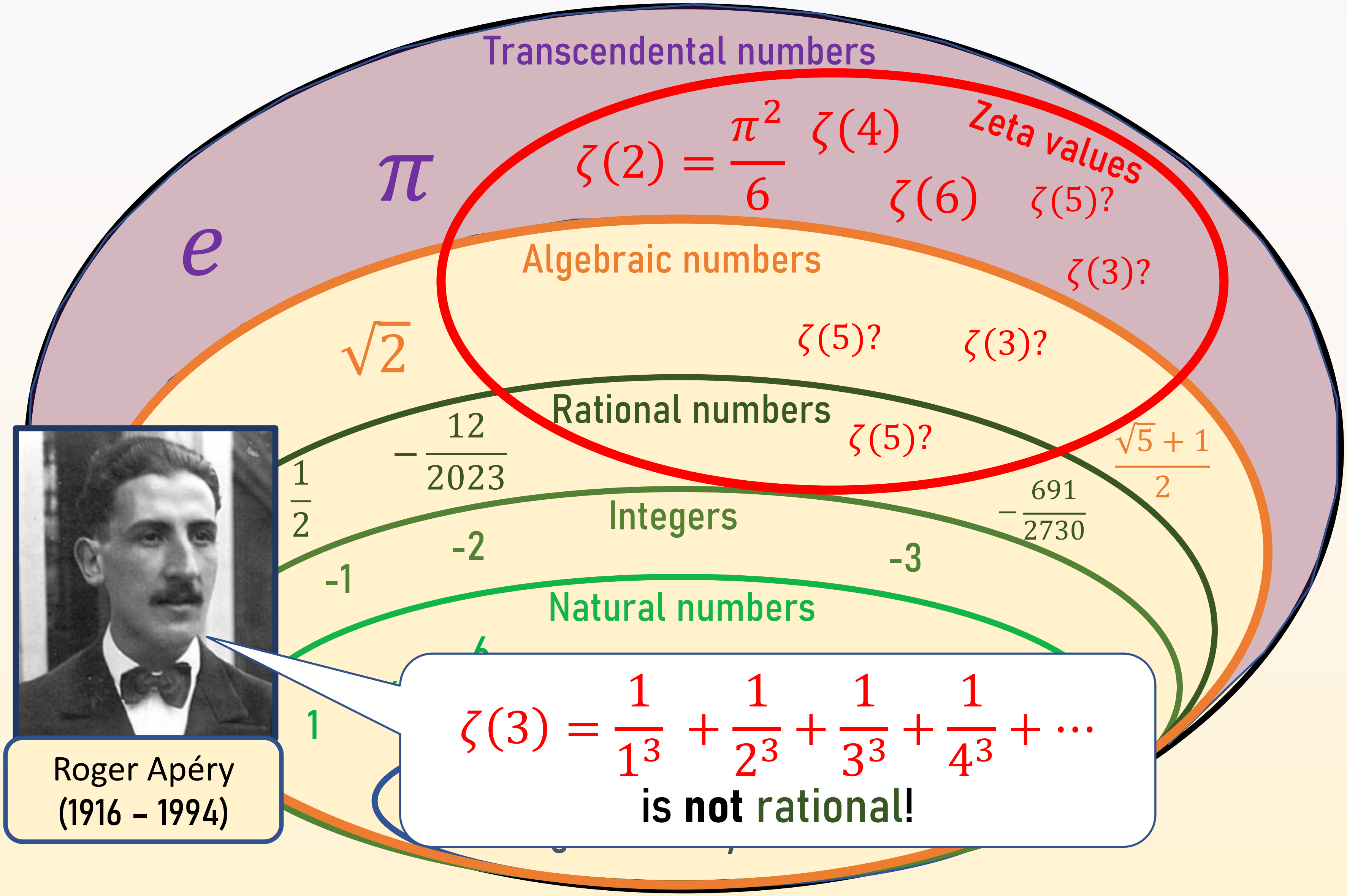
Euler's formulas imply:
If k is even then $\zeta(k)$ is transcendental

Bernhard Riemann
(1826 – 1866)

Conjecture:

$\zeta(k)$ is transcendental for all natural numbers k greater than 1

Classification of numbers



Multiple zeta values

The **Riemann zeta values** can also be written as:

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} = \sum_{n>0} \frac{1}{n^k} = \frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots$$

Sum over all integers n, which satisfy the condition $n > 0$

The **double zeta values** are defined by

$$\zeta(r,s) = \sum_{0<m<n} \frac{1}{m^r n^s} = \frac{1}{1^r 2^s} + \frac{1}{1^r 3^s} + \frac{1}{2^r 3^s} + \frac{1}{1^r 4^s} + \cdots$$

Sum over all integers m and n, which satisfy the condition $0 < m < n$

Multiple zeta values

the ramen...

For $k_1, k_2, \ldots, k_{r-1} \geq 1$ and $k_r \geq 2$ the **multiple zeta values** are defined by

$$\zeta(k_1, \ldots, k_r) = \sum_{0<n_1<\cdots<n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}$$

These numbers satisfy a lot of relations

Examples:

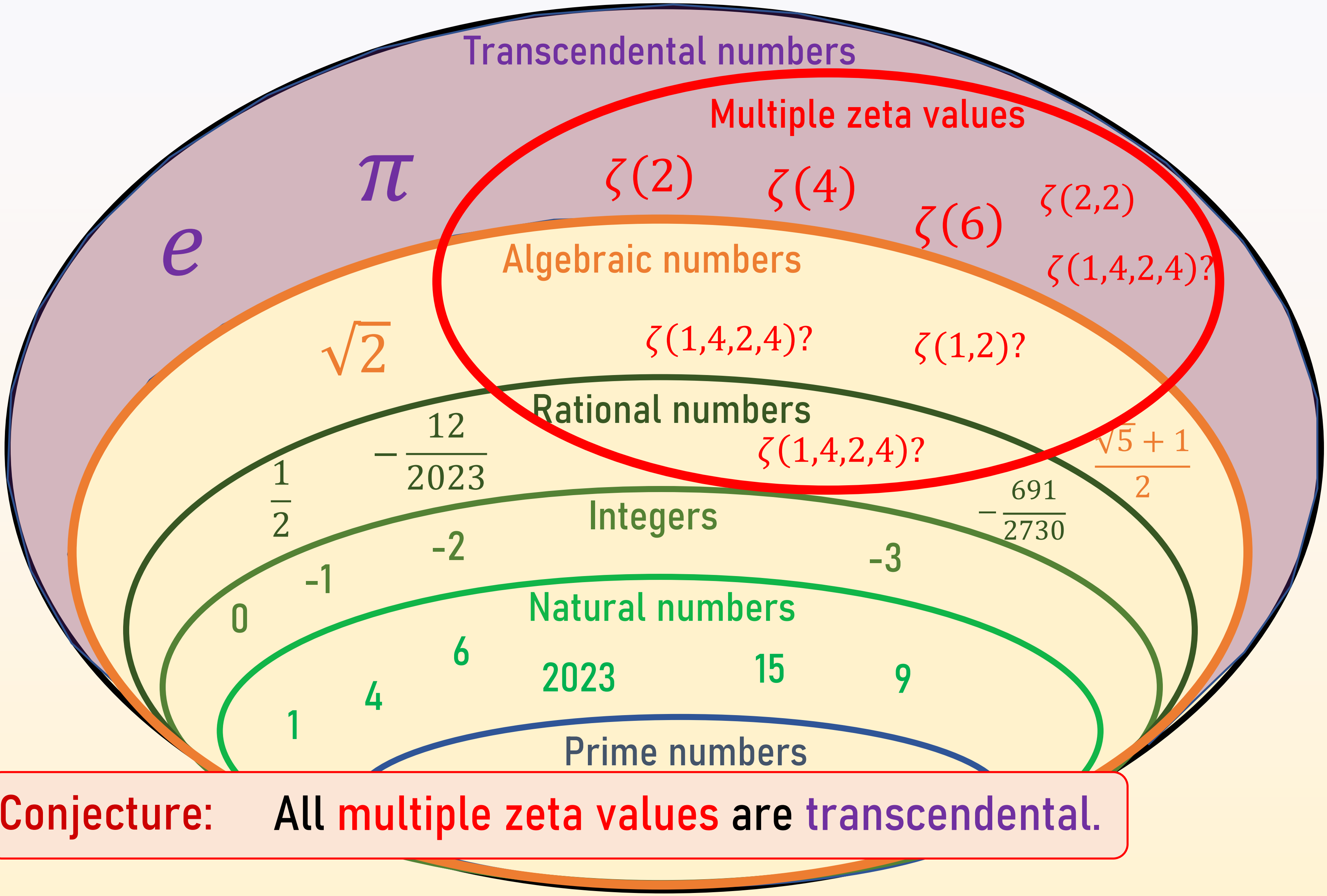
$$\zeta(3) = \zeta(1,2)$$

$$\frac{5197}{691}\zeta(12) = 168\zeta(7,5) + 150\zeta(5,7) + 28\zeta(3,9)$$

$$\zeta(\underbrace{2, \ldots, 2}_{n}) = \frac{\pi^{2n}}{(2n+1)!}$$

One of the goals is to understand all these relations

Classification of numbers



Conjecture: All multiple zeta values are transcendental.

Multiple zeta values

the ramen...

$$\zeta(k_1,\ldots,k_r)=\sum_{0<n_1<\cdots<n_r}\frac{1}{n_1^{k_1}\cdots n_r^{k_r}}$$

The product of two multiple zeta values is again a linear combination of multiple zeta values

$$\zeta(r)\zeta(s)=\sum_{m>0}\frac{1}{m^r}\sum_{n>0}\frac{1}{n^s}=\sum_{\substack{m>0\\n>0}}\frac{1}{m^r n^s}$$

$$=\sum_{0<m<n}\frac{1}{m^r n^s}+\sum_{0<n<m}\frac{1}{m^r n^s}+\sum_{0<m=n}\frac{1}{m^{r+s}}$$

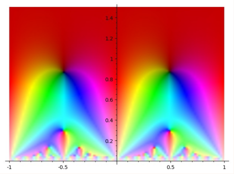
$$=\zeta(r,s)+\zeta(s,r)+\zeta(r+s)$$

Multiple zeta values

In my research I study the connections of
multiple zeta values and modular forms

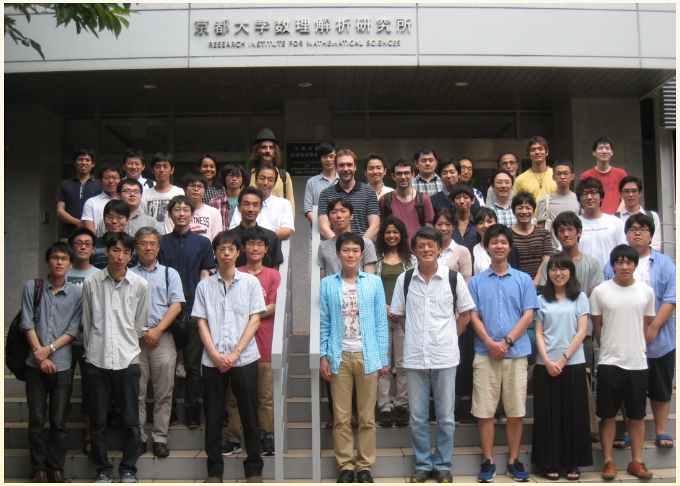
$$\zeta(k_1,\dots,k_r) = \sum_{0<n_1<\dots<n_r} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}}$$

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k f(\tau)$$



$$G_k(\tau) = \frac{1}{2} \sum_{\substack{m,n\in\mathbb{Z}\\(m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^k}$$

Both topics are popular in Japan



Thank you very much for your attention

… und frohe Weihnachtstage.