# Formal analogues of multiple zeta values and their variants

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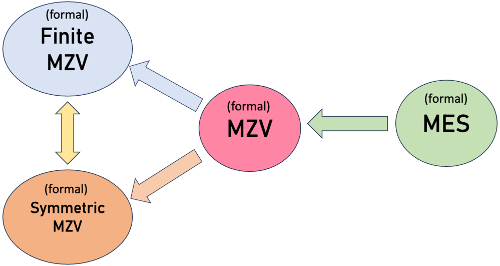


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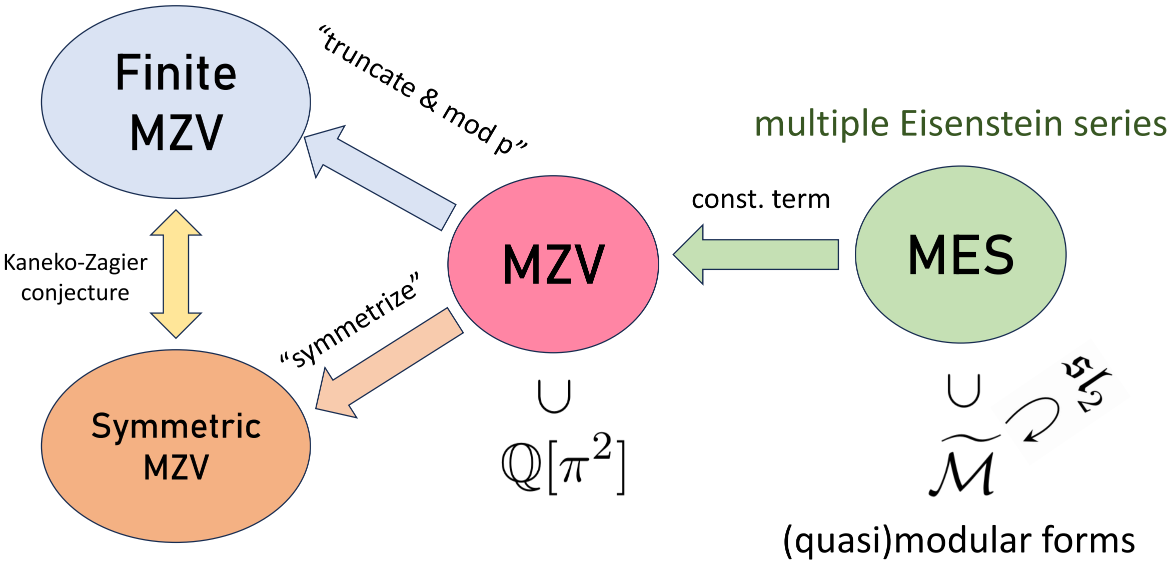
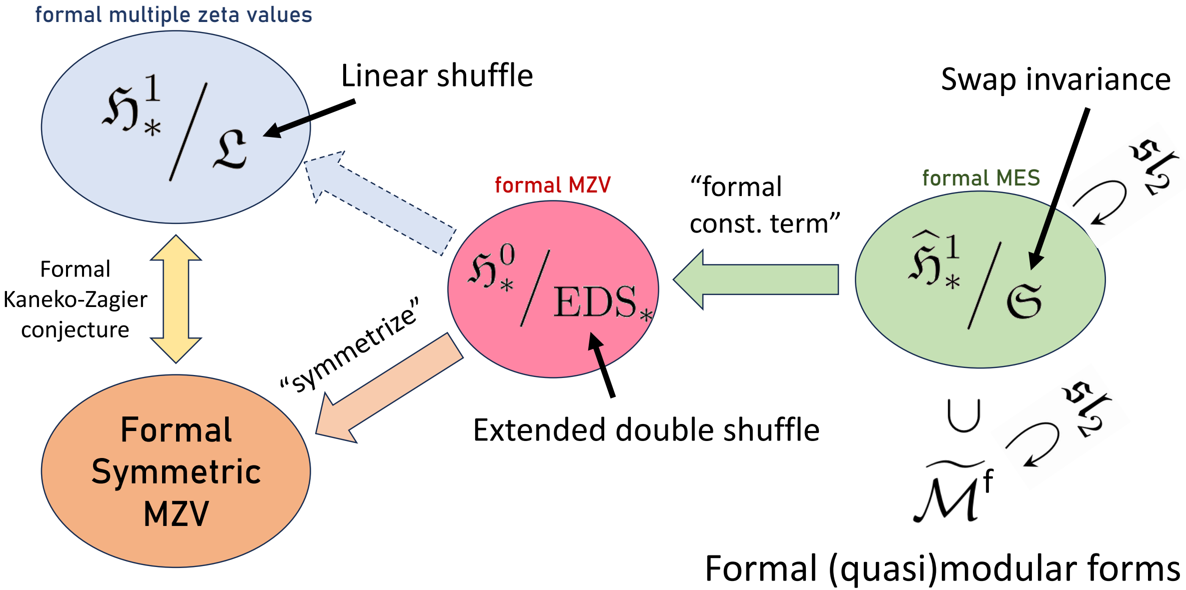
# Variants of multiple zeta values (MZV)



Formal version = Symbols / Relations

# Variants of formal multiple zeta values



**Definition (Same notation as in Prof. Zhonghua Li's talk)**

For an index $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r$ with $k_1 \geq 2, k_2, \dots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(\mathbf{k}) = \zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By $r$ we denote its **depth** and $k_1 + \dots + k_r$ will be called its **weight**.

- $\mathcal{Z}$ : $\mathbb{Q}$-algebra of MZVs
- $\mathcal{Z}_k$: $\mathbb{Q}$-vector space of MZVs of weight $k$.

- In the case $r = 1$ these are just the classical Riemann zeta values

$$\zeta(k) = \sum_{n>0} \frac{1}{n^k}\,, \qquad \zeta(2) = \frac{\pi^2}{6}\,, \qquad \zeta(3) \notin \mathbb{Q}\,, \qquad \zeta(4) = \frac{\pi^4}{90}\,, \quad \dots\,.$$

- MZVs were first studied by Euler ($r = 2$) and for general depth, they had their big comeback around 1990 due to their appearances in various areas of mathematics and physics.

MZVs can also be written as **iterated integrals**:

**Proposition**

The MZV $\zeta(k_1, \ldots, k_r)$ of weight $k = k_1 + \ldots + k_r$ can be written as an iterated integral

$$\zeta(k_1, \ldots, k_r) = \int_{1>t_1>\cdots>t_k>0} \omega_1(t_1) \cdots \omega_k(t_k) \,,$$

where

$$\omega_j(t) = \begin{cases} \frac{dt}{1-t} & \text{if } j \in \{k_1, k_1+k_2, \ldots, k_1+\cdots+k_r\} \\ \frac{dt}{t} & \text{else} \end{cases} .$$

**Example**

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5} .$$

There are two different ways to express the product of MZVs in terms of MZVs.

**Harmonic product (coming from the definition as iterated sums)**

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}} \\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}} \\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

**Shuffle product (coming from the expression as iterated integrals)**

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \int\ldots\cdot\int\ldots = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j,k_1+k_2-j)\,.$$

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

**Example**

$$\zeta(2,3)+3\zeta(3,2)+6\zeta(4,1) \overset{\text{shuffle}}{=} \zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g. $\zeta(2,1)=\zeta(3)$. These follow from regularizing the double shuffle relations and they are called **extended double shuffle relations**.

**Conjecture**

The extended double shuffle relations give all linear relations among MZV and

$$\mathcal{Z}=\bigoplus_{k\geq 0}\mathcal{Z}_k\,.$$

Next, we will introduce the basic algebraic setup to describe the harmonic and shuffle products.

**Definition**

- $\mathfrak{H} := \mathbb{Q}\langle x, y\rangle$,
- $\mathfrak{H}^0 := \mathbb{Q} + x\mathfrak{H}y \subset \mathfrak{H}^1 := \mathbb{Q} + \mathfrak{H}y \subset \mathfrak{H}$,
- **1**: empty word/monomial, $z_k := x^{k-1}y \in \mathfrak{H}^1 (k \geq 1)$,
- Weight: $\mathrm{wt}(z_{k_1} \cdots z_{k_r}) := k_1 + \cdots + k_r$,
- Reverse: $R : \mathfrak{H}^1 \to \mathfrak{H}^1$ be the linear map defined by $R(z_{k_1} \cdots z_{k_r}) := z_{k_r} \cdots z_{k_1}$.

We can view $z_k$ as individual "letters" and have $\mathfrak{H}^1 = \mathbb{Q}\langle z_1, z_2, \ldots\rangle$.

**Definition**

- $* : \mathfrak{H}^1 \otimes \mathfrak{H}^1 \to \mathfrak{H}^1$: **harmonic product**. For words $w, v \in \mathfrak{H}^1$ and $r, s \geq 1$,

$$\mathbf{1} * w = w = w * \mathbf{1},$$
$$z_r w * z_s v = z_r(w * z_s v) + z_s(z_r w * v) + z_{r+s}(w * v),$$

- $⧢ : \mathfrak{H} \otimes \mathfrak{H} \to \mathfrak{H}$: **shuffle product**. For words $w, v \in \mathfrak{H}$ and $a, b \in \{x, y\}$,

$$\mathbf{1} ⧢ w = w = w ⧢ \mathbf{1},$$
$$aw ⧢ bv = a(w ⧢ bv) + b(aw ⧢ v),$$

- For $\bullet \in \{*, ⧢\}$, we denote by $\mathfrak{H}^1_\bullet$ and $\mathfrak{H}^0_\bullet$ the algebras $(\mathfrak{H}^1, \bullet)$ and $(\mathfrak{H}^0, \bullet)$.

When viewing $\zeta$ as a linear map $\zeta : \mathfrak{H}^0 \to \mathbb{R}$, we can write the (finite) double shuffle relations for MZV as

$$\zeta(w)\zeta(v) = \zeta(w * v) = \zeta(w ⧢ v)$$

for all $w, v \in \mathfrak{H}^0$.

Let $\operatorname{reg}_* : \mathfrak{H}^1_* \cong \mathfrak{H}^0_*[z_1] \to \mathfrak{H}^0_*$ be the alg. hom. that is the identity on $\mathfrak{H}^0$ and maps $y = z_1$ to $0$.

**Definition**

- The space of **formal multiple zeta values** is defined by

$$\mathcal{Z}^f := \mathfrak{H}^0_* \big/ \operatorname{EDS}_*,$$

where $\operatorname{EDS}_*$ is the ideal in $\mathfrak{H}^0_*$ generated by $\operatorname{reg}_*(w * v - w ⧢ v)$ for all $w \in \mathfrak{H}^0, v \in \mathfrak{H}^1$.
- For $k_1 \geq 2$ and $k_2, ..., k_r \geq 1$, we denote the class of $z_{k_1} \cdots z_{k_r}$ in $\mathcal{Z}^f$ by $\zeta^f(k_1, ..., k_r)$.

**Conjecture (rephrased)**

The linear map

$$\mathcal{Z}^f \longrightarrow \mathcal{Z}$$

$$\zeta^f(k_1, \ldots, k_r) \longmapsto \zeta(k_1, \ldots, k_r)$$

is an algebra isomorphism.

**Definition (Kaneko-Zagier)**

For $k_1, \ldots, k_r \geq 1$, the **finite multiple zeta value** is defined as

$$\zeta_{\mathcal{A}}(k_1, \ldots, k_r) = \left( \sum_{p > m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right)_{p \text{ prime}} \in \mathcal{A},$$

where $\mathcal{A}$ is the $\mathbb{Q}$-algebra

$$\mathcal{A} = \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z} \Big/ \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}.$$

Again we can view these as an algebra homomorphism $\zeta_{\mathcal{A}} : \mathfrak{H}_*^1 \to \mathcal{A}$

$$\zeta_{\mathcal{A}}(z_{k_1} \cdots z_{k_r}) = \zeta_{\mathcal{A}}(k_1, \ldots, k_r).$$

**Definition**

- $\mathcal{Z}^{\mathcal{A}}$: $\mathbb{Q}$-vector space spanned by all finite MZVs,
- $\mathcal{Z}_k^{\mathcal{A}} \subset \mathcal{Z}^{\mathcal{A}}$: the subspace spanned by finite MZVs of weight $k$.

Finite MZVs satisfy the following relations.

**Harmonic product** For $w, v \in \mathfrak{H}^1$,

$$\zeta_{\mathcal{A}}(w) \cdot \zeta_{\mathcal{A}}(v) = \zeta_{\mathcal{A}}(w * v).$$

Therefore, $\mathcal{Z}^{\mathcal{A}}$ is also a $\mathbb{Q}$-algebra.

**Linear shuffle relation** For $w, v \in \mathfrak{H}^1$,

$$\zeta_{\mathcal{A}}(w ⧢ v) = (-1)^{\mathrm{wt}(w)} \zeta_{\mathcal{A}}(R(w)v).$$

When we set $v = \mathbf{1}$ in the second equation, we get the **reversal formula** $\zeta_{\mathcal{A}}(w) = (-1)^{\mathrm{wt}(w)} \zeta_{\mathcal{A}}(R(w))$.

**Conjecture (Kaneko-Zagier)**

- We have an $\mathbb{Q}$-algebra isomorphism

$$\begin{aligned} \mathcal{Z}^{\mathcal{A}} &\longrightarrow \mathcal{Z}/\pi^2\mathcal{Z} \\ \zeta_{\mathcal{A}}(k_1,\dots,k_r) &\longmapsto \zeta_{\mathcal{S}}(k_1,\dots,k_r)\,. \end{aligned}$$

  Here the $\zeta_{\mathcal{S}} \in \mathcal{Z}/\pi^2\mathcal{Z}$ are the symmetric MZVs (cf. Prof. Kaneko's & Prof. Zhaos's talk)
- All $\mathbb{Q}$-algebraic relations between finite MZVs can be deduced from the harmonic and linear shuffle relations.

- We do not even know if the above map is well-defined.
- The symmetric MZVs $\zeta_{\mathcal{S}}$ can be written in terms of MZVs, e.g.

$$\zeta_{\mathcal{S}}(k_1,k_2) \equiv (-1)^{k_1}\binom{k_1+k_2}{k_1}\zeta(k_1+k_2) \mod \pi^2\mathcal{Z}.$$

  The same relations holds by replacing $\mathcal{S}$ by $\mathcal{A}$ and $\zeta(k)$ by $Z(k) = \left(\frac{B_{p-k}}{k}\right)_p$.
- In general, the finite/symmetric MZVs of depth $r$ seem to "correspond" to MZVs of depth $r-1$.

**Definition**

- The algebra of **formal finite multiple zeta values** $\mathcal{F}$ is the $\mathbb{Q}$-algebra defined by

$$\mathcal{F} = \mathfrak{H}^1_* \big/ \mathfrak{L}$$

with $\mathfrak{L}$ be the $*$-ideal generated by $\{w \sqcup\!\sqcup\, v - (-1)^{\mathrm{wt}(w)} R(w) v \mid w, v \in \mathfrak{H}^1\}$.

- For $r \geq 1$ and $k_1, \dots, k_r \geq 1$, we denote the class of $z_{k_1} \cdots z_{k_r}$ by $\zeta^f_{\mathcal{A}}(k_1, \dots, k_r) \in \mathcal{F}$.

The linear map

$$\begin{aligned} \mathcal{F} &\longrightarrow \mathcal{Z}^{\mathcal{A}} \\ \zeta^f_{\mathcal{A}}(k_1, \dots, k_r) &\longmapsto \zeta_{\mathcal{A}}(k_1, \dots, k_r) \end{aligned}$$

is an algebra homomorphism (and conjecturally an isomorphism).

**Proposition (B.-Risan, 2024+)**

- For all $k \geq 1$ we have $\zeta^f_{\mathcal{A}}(k) = 0$. (already proven by Kaneko)
- For $k_1, k_2 \geq 1$ we have

$$\zeta^f_{\mathcal{A}}(k_1, k_2) = (-1)^{k_1} \binom{k_1 + k_2}{k_1} Z^f(k_1 + k_2),$$

where $Z^f(k) = -\frac{1}{k}\zeta^f_{\mathcal{A}}(k-1, 1)$.

**Proposition (Risan, 2024+)**

For odd $3 \leq k \leq 13$ we have $Z^f(k) \neq 0$.

**Definition**

- Let $\zeta^f : \mathfrak{H}^0 \to \mathcal{Z}^f$ be the linear map such that $\zeta^f(z_{k_1} \cdots z_{k_r}) = \zeta^f(k_1, \ldots, k_r)$.
- Extend the map $\zeta^f$ to an algebra homomorphisms $\zeta^{f,\bullet} : \mathfrak{H}^1_\bullet \to \mathcal{Z}^f[T]$ for $\bullet \in \{*, ш\}$ with $z_1 \mapsto T$.
- For $\bullet \in \{*, ш\}$, we define the linear map $\zeta_S^{f,\bullet} : \mathfrak{H}^1 \to \mathcal{Z}^f[T]$ by

$$\zeta_S^{f,\bullet} := (\zeta^{f,\bullet} \circ W \circ R) \star \zeta^{f,\bullet},$$

where $W : \mathfrak{H}^1 \to \mathfrak{H}^1$ denote the linear map defined by

$$W(w) = (-1)^{\mathrm{wt}(w)} w$$

for word $w \in \mathfrak{H}^1$ and $\star$ denotes the convolution product with respect to the deconcatenation coproduct.

Again we define for $k_1, \dots, k_r \geq 1$

$$\zeta_S^{f,\bullet}(k_1, \dots, k_r) = \zeta_S^{f,\bullet}(z_{k_1} \cdots z_{k_r}).$$

With this we get similar to the usual definition of classical $\zeta_S^{\bullet}$

$$\zeta_S^{f,\bullet}(k_1, \dots, k_r) = \sum_{j=0}^{r} (-1)^{k_1+\cdots+k_j} \zeta^{f,\bullet}(k_j, \dots, k_1) \zeta^{f,\bullet}(k_{j+1}, \dots, k_r).$$

Similar to the symmetric multiple zeta values, we have the following theorems.

**Proposition (B.-Risan 2024+)**

- The polynomial $\zeta_S^{f,\bullet}(z_{k_1} \cdots z_{k_r})$ is independent of T.
- The map $\zeta_S^{f,*} : \mathfrak{H}_*^1 \to \mathcal{Z}^f$ is an algebra homomorphism.

Proof sketch: $R$ and $W$ are algebra homomorphisms with respect to $*$.

The following results are formal versions of the analogue statements for classical finite MZV by Kaneko-Zagier.

**Proposition (B.-Risan 2024+)**

For any $w, v \in \mathfrak{H}^1$, we have

$$\zeta_S^{f,ш}(w \sqcup\!\sqcup v) = (-1)^{\mathrm{wt}(w)} \zeta_S^{f,ш}(R(w)v).$$

**Proposition (B.-Risan 2024+)**

For any $r \geq 0$ and $k_1, \ldots, k_r \geq 1$,

$$\zeta_S^{f,*}(k_1, \ldots k_r) \equiv \zeta_S^{f,ш}(k_1, \ldots k_r) \pmod{\zeta^f(2)\mathcal{Z}^f}.$$

**Definition**

The **formal symmetric MZV** $\zeta_S^f(k_1, \ldots, k_r)$ is the class of $\zeta_S^{f,*}(k_1, \ldots, k_r)$ in $\mathcal{Z}^f \,/\, \zeta^f(2)\mathcal{Z}^f$.

Again we view these as a map $\zeta_S^f : \mathfrak{H}^1 \to \mathcal{Z}^f \,/\, \zeta^f(2)\mathcal{Z}^f$.

**Corollary**

For any $w, v \in \mathfrak{H}^1$, we have

$$\zeta_S^f(w * v) = \zeta_S^f(w)\zeta_S^f(v),$$
$$\zeta_S^f(w \sqcup\!\sqcup v) = (-1)^{\mathrm{wt}(w)}\zeta_S^f(R(w)v).$$

**Corollary**

The map

$$\mathcal{F} \longrightarrow \mathcal{Z}^f \Big/ \zeta^f(2)\mathcal{Z}^f$$
$$\zeta_{\mathcal{A}}^f(k_1, \dots, k_r) \longmapsto \zeta_S^f(k_1, \dots, k_r)$$

is an algebra homomorphism (which conjecturally is an isomorphism).

Let $\mathrm{Fil}_d^{\mathrm{dep}}\,\mathcal{F}_k$ be the subspace of formal finite MZV of weight $k$ and depth at most $d$.

**Theorem (Weak parity)**

*Let $k, d \geq 1$. If $k$ and $d$ have the same parity then*

$$\mathrm{Fil}_d^{\mathrm{dep}}\,\mathcal{F}_k \subset \mathrm{Fil}_{d-1}^{\mathrm{dep}}\,\mathcal{F}_k + \sum_{\substack{k_1+k_2=k\\ d_1+d_2=d\\ d_1,d_2\geq 2}} \mathrm{Fil}_{d_1}^{\mathrm{dep}}\,\mathcal{F}_{k_1} \cdot \mathrm{Fil}_{d_2}^{\mathrm{dep}}\,\mathcal{F}_{k_2}.$$

**Proof idea** This is a direct consequence of the antipode relation

$$\sum_{j=0}^{d}(-1)^j \zeta_{\mathcal{A}}^f(k_j,\ldots,k_1)\zeta_{\mathcal{A}}^f(k_{j+1},\ldots,k_d) \equiv 0 \mod \mathrm{Fil}_{d-1}^{\mathrm{dep}}\,\mathcal{F}_k, \quad (k = k_1+\cdots+k_d)$$

the reversal formula $\zeta_{\mathcal{A}}^f(k_1,\ldots,k_d) = (-1)^{k_1+\cdots+k_d}\zeta_{\mathcal{A}}^f(k_d,\ldots,k_1)$, and the fact that $\zeta_{\mathcal{A}}^f(k) = 0$.

**Conjecture (Strong parity)**

For $k, d \geq 1$ where $k$ and $d$ have the same parity, we have

$$\mathrm{Fil}^{\mathrm{dep}}_{d}\, \mathcal{F}_k \subset \mathrm{Fil}^{\mathrm{dep}}_{d-1}\, \mathcal{F}_k.$$

**Proposition (Risan 2024+)**

For $k \geq 1$ and $1 \leq d \leq 4$, the above conjecture is true.

**Proof idea**

- The statement for $1 \leq d \leq 3$ follows directly from the weak parity.
- The relations in $\mathrm{Fil}^{\mathrm{dep}}_{4}\, \mathcal{F}_k$ can be described as actions by the group ring $\mathbb{Z}[\mathrm{GL}_4(\mathbb{Z})]$. Then, the proof is done by computer-based calculations .

For $\tau \in \mathbb{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$ define on $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2)\,.$$

**Definition (Gangl-Kaneko-Zagier $r = 2$, 2006)**

For $k_1, \dots, k_r \geq 2$ the **multiple Eisenstein series** are defined by

$$\mathbb{G}(k_1, \dots, k_r; \tau) = \mathbb{G}(k_1, \dots, k_r) = \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}}\,.$$

**Theorem (Gangl-Kaneko-Zagier 2006 $r = 2$, B. 2012 $r \geq 2$)**

The multiple Eisenstein series have a Fourier expansion of the form

$$\mathbb{G}(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r) + \sum_{n \geq 1} a_n^{k_1,\dots,k_r} q^n, \qquad (q = e^{2\pi i \tau}),$$

with (explicit) $a_n^{k_1,\dots,k_r} \in \mathcal{Z}[\pi i]$.

As an extension of the $\mathfrak{H}^1 = \mathbb{Q}\langle z_1, z_2, \dots \rangle$ we define

$$\widehat{\mathfrak{H}}^1 = \mathbb{Q} + \langle z_{k_1} \cdots z_{k_r} \mid r \geq 1, k_1 \geq 1, k_2, \dots, k_r \geq 0 \rangle_{\mathbb{Q}} \quad \subset \quad \mathbb{Q}\langle z_0, z_1, z_2, \dots \rangle.$$

($\widehat{\mathfrak{H}}^1$= words in $z_0, z_1, z_2 \dots$ not starting in $z_0$)

We extend the **harmonic product** to $\widehat{\mathfrak{H}}^1 \otimes \widehat{\mathfrak{H}}^1 \to \widehat{\mathfrak{H}}^1$ for words $w, v \in \widehat{\mathfrak{H}}^1$ and $r, s \geq 0$ by

$$\mathbf{1} * w = w = w * \mathbf{1},$$
$$z_r w * z_s v = z_r (w * z_s v) + z_s (z_r w * v) + \delta_{r \cdot s \neq 0} \, z_{r+s} (w * v),$$

and obtain $\mathbb{Q}$-algebras $\mathfrak{H}^1_* \subset \widehat{\mathfrak{H}}^1_*$.

Define the **swap** $\sigma$ as the linear map given for $k_1, \dots, k_r, m_1, \dots, m_r \geq 1$ by

$$\sigma : \widehat{\mathfrak{H}}^1 \longrightarrow \widehat{\mathfrak{H}}^1,$$
$$z_{k_1} z_0^{m_1 - 1} z_{k_2} z_0^{m_2 - 1} \dots z_{k_r} z_0^{m_r - 1} \longmapsto z_{m_r} z_0^{k_r - 1} \dots z_{m_1} z_0^{k_1 - 1}.$$

(cf. with the usual duality of MZV)

**Definition**

- The algebra of **formal multiple Eisenstein series** $\mathcal{G}^f$ is the $\mathbb{Q}$-algebra defined by

$$\mathcal{G}^f = \widehat{\mathfrak{H}}^1_* \Big/ \mathfrak{S}$$

  where $\mathfrak{S}$ is the $*$-ideal generated by $\{w - \sigma(w) \mid w \in \widehat{\mathfrak{H}}^1\}$.
- By $G^f(k_1, \dots, k_r)$ we denote the class of $z_{k_1} \cdots z_{k_r}$ for $k_1 \geq 1, k_2, \dots, k_r \geq 0$.

**Conjecture:** $G^f(k_1, \dots, k_r)$ satisfy the same relations as $\mathbb{G}(k_1, \dots, k_r)$ if $k_1, \dots, k_r \geq 2$.

**Definition**

An algebra $A$ is called **$\mathfrak{sl}_2$-algebra** if there exists a Lie-algebra homomorphism $\mathfrak{sl}_2 \to \mathrm{Der}(A)$.

In other words, there exist derivations $D, W, \delta \in \mathrm{Der}(A)$ satisfying the commutator relations

$$[W, D] = 2D, \quad [W, \delta] = -2\delta, \quad [\delta, D] = W.$$

**Example** The algebra of **quasimodular forms** $\widetilde{\mathcal{M}} = \mathbb{Q}[G(2), G(4), G(6)]$ with

$$D = q\frac{d}{dq}, \qquad W(G(k)) = kG(k), \qquad \delta(G(2)) = -\frac{1}{2}, \quad \delta(G(4)) = \delta(G(6)) = 0$$

is an $\mathfrak{sl}_2$ algebra. Here the Eisenstein series $G(k)$ are given for $k \geq 2$ by

$$G(k) = (-2\pi i)^{-k}\mathbb{G}(k) = -\frac{B_k}{2k!} + \frac{1}{(k-1)!}\sum_{m,n\geq 1} m^{k-1}\, q^{mn}, \qquad (B_k = k\text{th Bernoulli number}).$$

# ③ Formal Multiple Eisenstein series - Results

**Theorem (B.-van-Ittersum, 2023)**

*There exist explicit derivations* $W, D, \delta$ *on* $\mathcal{G}^f$ *such that*

- $\mathcal{G}^f$ *is an* $\mathfrak{sl}_2$*-algebra;*
- *the subalgebra* $\mathbb{Q}[G^f(2), G^f(4), G^f(6)] \subset \mathcal{G}^f$ *is isomorphic to* $\widetilde{\mathcal{M}}$ *as an* $\mathfrak{sl}_2$*-algebra.*

**Theorem (B.-van-Ittersum, 2023)**

*There exists a surjective algebra homomorphism (The "formal projection to the constant term")*

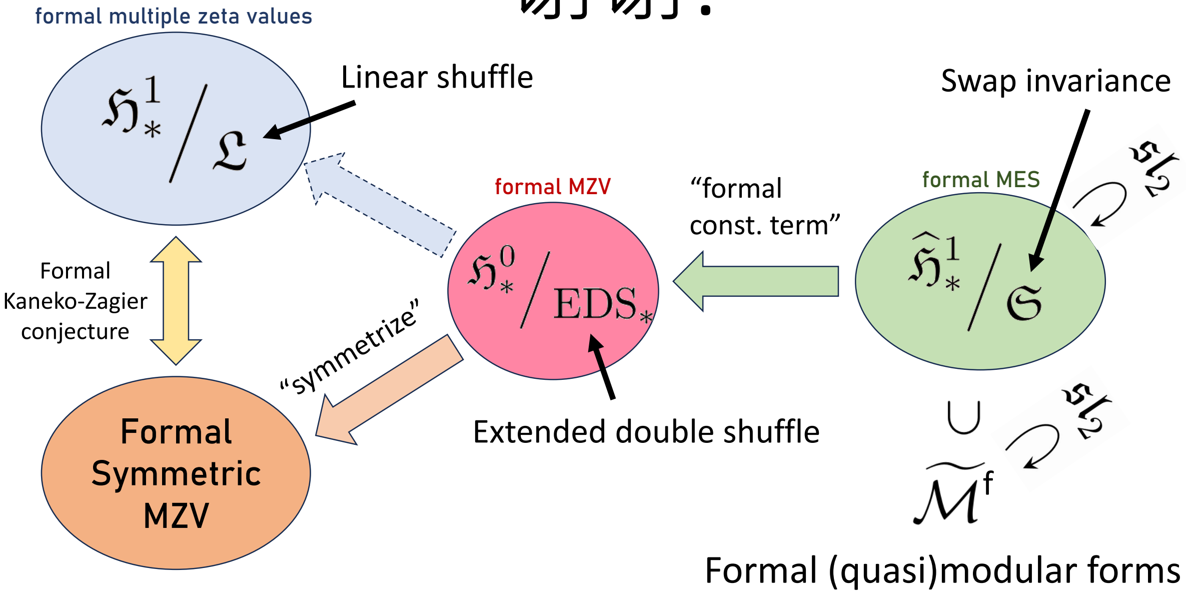
$$\pi : \mathcal{G}^f \to \mathcal{Z}^f,$$

*with* $\pi(G^f(k_1, \ldots, k_r)) = \zeta^f(k_1, \ldots, k_r)$*. The kernel of* $\pi$ *can be described explicitly.*

**Theorem (B.-Burmester, 2023)**

*There exists an algebra homomorphism* $\mathcal{G}^f \to \mathbb{Q}[[q]]$ *with* $G^f(k) \mapsto G(k)$*.*

# 谢谢！


formal multiple zeta values
$\mathfrak{H}^1_* / \mathfrak{L}$
Linear shuffle
Formal Kaneko-Zagier conjecture
Formal Symmetric MZV
"symmetrize"
formal MZV
$\mathfrak{H}^0_* / \mathrm{EDS}_*$
Extended double shuffle
"formal const. term"
formal MES
$\widehat{\mathfrak{H}}^1_* / \mathfrak{S}$
Swap invariance
$\mathfrak{sl}_2$
$\cup$
$\widetilde{\mathcal{M}}^{\mathrm{f}}$
$\mathfrak{sl}_2$
Formal (quasi)modular forms

**Example** As a more complicated example, the following gives an expression of a depth 4 formal finite MZV in terms of depth 3 formal finite MZVs.

$$\begin{aligned}
\zeta^f_{\mathcal{A}}(2,3,1,4) = {} & 81\zeta^f_{\mathcal{A}}(1,1,8) - \frac{497}{4}\zeta^f_{\mathcal{A}}(1,2,7) - \frac{483}{4}\zeta^f_{\mathcal{A}}(1,3,6) - 63\zeta^f_{\mathcal{A}}(1,4,5) \\
& - 22\zeta^f_{\mathcal{A}}(1,5,4) - \frac{5}{2}\zeta^f_{\mathcal{A}}(1,6,3) - 68\zeta^f_{\mathcal{A}}(1,7,2) + \frac{507}{4}\zeta^f_{\mathcal{A}}(1,8,1) \\
& - \frac{277}{4}\zeta^f_{\mathcal{A}}(2,1,7) - \frac{333}{4}\zeta^f_{\mathcal{A}}(2,2,6) - \frac{167}{4}\zeta^f_{\mathcal{A}}(2,3,5) - \frac{59}{4}\zeta^f_{\mathcal{A}}(2,4,4) \\
& + 2\zeta^f_{\mathcal{A}}(2,5,3) + \frac{27}{4}\zeta^f_{\mathcal{A}}(2,6,2) + 126\zeta^f_{\mathcal{A}}(2,7,1) - \frac{175}{4}\zeta^f_{\mathcal{A}}(3,1,6) \\
& - \frac{33}{2}\zeta^f_{\mathcal{A}}(3,2,5) - 5\zeta^f_{\mathcal{A}}(3,3,4) - \frac{5}{4}\zeta^f_{\mathcal{A}}(3,4,3) - \frac{13}{2}\zeta^f_{\mathcal{A}}(4,1,5) + \zeta^f_{\mathcal{A}}(4,2,4).
\end{aligned}$$

**Definition**

For $k_1, \ldots, k_r \geq 1$ we define the $q$-series $g(k_1, \ldots, k_r) \in \mathbb{Q}[[q]]$ by

$$g(k_1, \ldots, k_r) = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \frac{n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \cdots + m_r n_r} .$$

In the case $r = 1$ these are the generating series of divisor-sums $\sigma_{k-1}(n) = \sum_{d|n} n^{k-1}$

$$g(k) = \sum_{m,n>0} \frac{n^{k-1}}{(k-1)!} q^{mn} = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n ,$$

and they can be viewed as $q$**-analogues of multiple zeta values**, since for $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1 - q)^{k_1 + \cdots + k_r} g(k_1, \ldots, k_r) = \zeta(k_1, \ldots, k_r) .$$

$$\hat{g}(k_1,\dots,k_r) := (-2\pi i)^{k_1+\dots+k_r} g(k_1,\dots,k_r) \in \mathbb{Q}[\pi i][\![q]\!]\,.$$

**Theorem (Gangl-Kaneko-Zagier 2006 ($r=2$), B. 2012 ($r\geq 2$))**

*For $k_1,\dots,k_r \geq 2$ there exist explicit $\alpha^{k_1,\dots,k_r}_{l_1,\dots,l_r,j} \in \mathbb{Z}$, such that for $q = e^{2\pi i\tau}$ we have*

$$\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{\substack{0<j<r \\ l_1+\dots+l_r=k_1+\dots+k_r}} \alpha^{k_1,\dots,k_r}_{l_1,\dots,l_r,j}\, \zeta(l_1,\dots,l_j)\hat{g}(l_{j+1},\dots,l_r) + \hat{g}(k_1,\dots,k_r)\,.$$

*In particular,* $\mathbb{G}(k_1,\dots,k_r) = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_{k_1,\dots,k_r}(n) q^n$ *for some* $a_{k_1,\dots,k_r}(n) \in \mathcal{Z}[\pi i]$.

**Examples**

$$\mathbb{G}(k) = \zeta(k) + \hat{g}(k)\,,$$

$$\mathbb{G}(3,2) = \zeta(3,2) + 3\zeta(3)\hat{g}(2) + 2\zeta(2)\hat{g}(3) + \hat{g}(3,2)\,.$$