Homework 2

Deadline: 14th November (23:55 JST), 2021

Exercise 5. We saw that in $R = \mathbb{Z}[\sqrt{-5}]$ we have the non-unique factorization of 6 into irreducible elements as $6 = 2 \cdot 3 = (1+\sqrt{-5}) \cdot (1-\sqrt{-5})$. Find prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3 \subset R$ such that the ideals generated by these elements can be written as

$$(2) = \mathfrak{p}_1^2\,, \quad (3) = \mathfrak{p}_2\mathfrak{p}_3\,, \quad (1+\sqrt{-5}) = \mathfrak{p}_1\mathfrak{p}_2\,, \quad (1-\sqrt{-5}) = \mathfrak{p}_1\mathfrak{p}_3$$

and conclude $(6) = \mathfrak{p}_1^2\mathfrak{p}_2\mathfrak{p}_3$.

Exercise 6.

(i) Let R be a commutative unitary ring and M a R-module. Show that the following two statements are equivalent definitions for M being noetherian

(a) All submodules of M are finitely generated.

(b) Any sequence $M_1 \subset M_2 \subset M_3 \subset \dots$ of submodules of M eventually stabilizes, i.e. there exists some n such that $M_n = M_{n+1} = M_{n+2} = \dots$.

(ii) Let R be a noetherian ring and M a R-module. Show that M is a noetherian module if and only if M is finitely generated.

Exercise 7. Let $d \neq 1, 0$ be a square-free integer. Show that the ring of integers of $K = \mathbb{Q}(\sqrt{d})$ is

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{d}] & d \equiv 2,3 \mod 4 \\ \mathbb{Z}\left[\frac{\sqrt{d}+1}{2}\right] & d \equiv 1 \mod 4 \end{cases},$$

i.e. show that the above set give exactly those elements in K which are integral over $\mathbb{Z}$.

Version: October 28, 2021