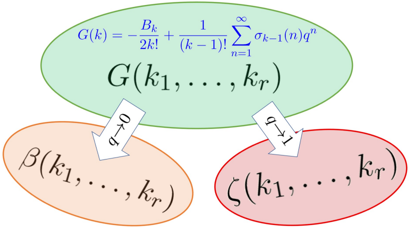
Connecting modular forms and multiple zeta values via combinatorial multiple Eisenstein series

Henrik Bachmann

Nagoya University



j.w.: A. Burmester, J.W. van Ittersum & N. Matthes, U. Kühn & N. Matthes (arXiv:2109.04267)

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www.henrikbachmann.com

	Numbers	Functions / q-series
"single" objects	Riemann zeta values $$\zeta(k) = \sum_{m>0} \frac{1}{m^k}$$	Eisenstein series
"multiple" objects	Multiple zeta values	Multiple Eisenstein series
relations object point of view	Double shuffle relations	
Mould / generating series point of view	Symmetril & Symmetral	Symmetril & Swap invariant

Definition

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ we define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs

MZVs can also be written as **iterated integrals**, e.g.

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}\,.$$

There are two different ways to express the product of MZV in terms of MZV.

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}}\\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j,k_1+k_2-j)\,.$$

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)$$
$$\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\zeta(2,1)=\zeta(3), \qquad \zeta(2)^2=\frac{5}{2}\zeta(4)\,.$$

These follow from regularizing the double shuffle relations: $\rightsquigarrow$ **extended double shuffle relations**.

- Modular forms are holomorphic functions on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$ satisfying certain functional equation (see bonus slides).
- They appear in various areas of mathematics and play an essential role in number theory.

Remark

Multiple zeta values have various different (and partially still conjectured) connection to modular forms, e.g.

- Broadhurst-Kreimer conjecture (see bonus slides),
- Exotic relations.

Riemann zeta values also appear in the Fourier expansion of the **Eisenstein series** defined for even $k \geq 4$ by

$$\mathbb{G}_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor sum, $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau}$.

For $\tau \in \mathbb{H}$ we define on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ the **order** $\succ$ by

$$m_1\tau + n_1 \succ m_2\tau + n_2 \quad :\Leftrightarrow \quad (m_1 > m_2) \text{ or } (m_1 = m_2 \text{ and } n_1 > n_2)\,.$$

Definition

For integers $k_1 \geq 3, k_2, \dots, k_r \geq 2$, we define the **multiple Eisenstein series** by

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}\,.$$

- These are holomorphic functions on the upper-half plane $\mathbb{H}$, but in general they are not modular.
- The product of multiple Eisenstein series can also be express by the **harmonic product** formula, e.g.

$$\mathbb{G}_4(\tau) \cdot \mathbb{G}_3(\tau) = \mathbb{G}_{4,3}(\tau) + \mathbb{G}_{3,4}(\tau) + \mathbb{G}_7(\tau)\,.$$

Definition

For $k_1, \dots, k_r \geq 1$ we define the q-series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ n_1, \dots, n_r > 0}} \frac{n_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{n_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 n_1 + \dots + m_r n_r} .$$

In the case $r = 1$ these are the generating series of divisor-sums $\sigma_{k-1}(n) = \sum_{d|n} n^{k-1}$

$$g(k) = \sum_{m,n > 0} \frac{n^{k-1}}{(k-1)!} q^{mn} = \frac{1}{(k-1)!} \sum_{n > 0} \sigma_{k-1}(n) q^n ,$$

and they can be viewed as ***q*-analogues of multiple zeta values**, since for $k_1 \geq 2, k_2, \dots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1 - q)^{k_1 + \dots + k_r} g(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r) .$$

Theorem (Gangl-Kaneko-Zagier 2006 ($r=2$), B. 2012 ($r\geq 2$))

The multiple Eisenstein series $\mathbb{G}_{k_1,\dots,k_r}(\tau)$ have a Fourier expansion of the form

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_n q^n \qquad \left(q = e^{2\pi i\tau}\right)$$

with $a_n \in \mathcal{Z}[2\pi i]$. Moreover, they can be written explicitly as a $\mathcal{Z}[2\pi i]$-linear combination of the q-series g.

Examples

$$\mathbb{G}_k(\tau) = \zeta(k) + (-2\pi i)^k g(k)\,,$$

$$\mathbb{G}_{3,2}(\tau) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3,2)\,.$$

(B.-Tasaka 2017): The Fourier expansion of $\mathbb{G}$ is related to the Goncharov coproduct (see bonus slides).

Definition

For $k_1, \dots, k_r \geq 1$ with $k_1, k_r \geq 2$ and $\tau \in \mathbb{H}$ define the **multitangent function** by

$$\Psi_{k_1,\dots,k_r}(\tau) := \sum_{\substack{n_1 > \dots > n_r \\ n_i \in \mathbb{Z}}} \frac{1}{(\tau + n_1)^{k_1} \cdots (\tau + n_r)^{k_r}}.$$

Theorem (Bouillot 2011, B. 2012)

For $k_1, \dots, k_r \geq 1$ with $k_1, k_r \geq 2$ and $K = k_1 + \dots + k_r$ the multitangent function can be written as

$$\Psi_{k_1,\dots,k_r}(\tau) = \sum_{j=2}^{K} \alpha_{K-j} \Psi_j(\tau), \qquad (\alpha_{K-j} \in \mathcal{Z}_{K-j}).$$

We obtain the following "mould product" decomposition

$$\mathbb{G}_{k_1,k_2}(\tau) = \zeta(k_1,k_2) + \sum_{m>0} \Psi_{k_1}(m\tau)\zeta(k_2) + \sum_{m>0} \Psi_{k_1,k_2}(m\tau) + \sum_{m_1 > m_2 > 0} \Psi_{k_1}(m_1\tau)\Psi_{k_2}(m_2\tau).$$

Calculation of the Fourier expansion of multiple Eisenstein series

Multiple Eisenstein series

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}$$

$$\mathbb{G}_{k_1,k_2}(\tau) = \zeta(k_1,k_2) + \sum_{m>0} \Psi_{k_1}(m\tau)\zeta(k_2) + \sum_{m>0} \Psi_{k_1,k_2}(m\tau) + \sum_{m_1>m_2>0} \Psi_{k_1}(m_1\tau)\Psi_{k_2}(m_2\tau)$$

$$\Psi_{k_1,\dots,k_r}(\tau) := \sum_{\substack{n_1>\cdots>n_r \\ n_i \in \mathbb{Z}}} \frac{1}{(\tau+n_1)^{k_1} \cdots (\tau+n_r)^{k_r}}.$$

mould product decomposition

MZV

$$\zeta(k_1,\dots,k_r)$$

Sums of multitangent functions

$$\sum_{m>0} \Psi_{k_1,k_2}(m\tau) + \sum_{m_1>m_2>0} \Psi_{k_1}(m_1\tau)\Psi_{k_2}(m_2\tau)$$

Reduction multitangent to monotangent

$$\Psi_{k_1,\dots,k_r}(\tau) = \sum_{j=2}^{K} \alpha_{K-j}\Psi_j(\tau)$$

q-MZV (sums of monotangent functions)

$$g(k_1,\dots,k_r) = (-2\pi i)^{-(k_1+\cdots+k_r)} \sum_{m_1>\cdots>m_r>0} \Psi_{k_1}(m_1\tau)\dots\Psi_{k_r}(m_r\tau)$$

We saw that multiple zeta values satisfy various relations. For example,

$$\zeta(2)^2 = \frac{5}{2}\zeta(4)\,,$$
$$\zeta(5) = 2\zeta(3,2) + 6\zeta(4,1)\,.$$

Question

Do multiple Eisenstein series satisfy these relations?

The first relation is clearly not satisfied, since setting $G_k = (-2\pi i)^{-k}\mathbb{G}_k$ we have

$$G_2^2 = \frac{5}{2}G_4 - \frac{1}{2}q\frac{d}{dq}G_2\,.$$

The second relation can not be satisfied since $\mathbb{G}_{4,1}$ is not defined.

② Multiple Eisenstein series - Extended definitions

There are different ways to extend the definition of $\mathbb{G}_{k_1,\dots,k_r}$ to $k_1 \geq 2, k_2, \dots, k_r \geq 1$

- Formal double zeta space realization $\mathbb{G}_{r,s}$ (Gangl-Kaneko-Zagier, 2006)

$$\mathbb{G}_{k_1} \cdot \mathbb{G}_{k_2} + (\delta_{k_1,2} + \delta_{k_2,2}) \frac{\mathbb{G}'_{k_1+k_2-2}}{2(k_1+k_2-2)} = \mathbb{G}_{k_1,k_2} + \mathbb{G}_{k_2,k_1} + \mathbb{G}_{k_1+k_2}$$
$$= \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \mathbb{G}_{j,k_1+k_2-j}\,, \qquad (k_1+k_2 \geq 3)\,.$$

- Shuffle regularized multiple Eisenstein series $\mathbb{G}^{ш}_{k_1,\dots,k_r}$ (B.-Tasaka, 2017).
- Harmonic regularized multiple Eisenstein series $\mathbb{G}^{*}_{k_1,\dots,k_r}$ (B., 2019).

Observation & Motivating question

- No version of these objects satisfy the double shuffle relations for all indices/weights.
- The derivative is always somewhere as an extra term.
- **What is a "natural" family of relations for multiple Eisenstein series (and their derivatives)?**
- In the following we will propose an algebraic approach using generating series.

	Numbers	Functions / q-series
"single" objects	Riemann zeta values $\zeta(k) = \sum_{m>0} \frac{1}{m^k}$	Eisenstein series $\mathbb{G}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ $(q = e^{2\pi i \tau})$
"multiple" objects	Multiple zeta values $\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$	Multiple Eisenstein series $\mathbb{G}_{k_1,\ldots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}} = \zeta(k_1, \ldots, k_r) + \sum_{n>0} a_n q^n$
relations object point of view	Double shuffle relations $\zeta(2) \cdot \zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$ $= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$ $\zeta(2)^2 = \frac{5}{2}\zeta(4)$	$G_2^2 = \frac{5}{2} G_4 - \frac{1}{2} q \frac{d}{dq} G_2$ $G_k = (-2\pi i)^{-k} \mathbb{G}_k$ Symmetril & Swap invariant
Mould / generating series point of view	Symmetril & Symmetral	

- L: countable set (set of **letters**).
- $\diamond$: commutative and associative product on $\mathbb{Q}L$.
- **word**: monic monomial in the non-commutative polynomial ring $\mathbb{Q}\langle L\rangle$. ($\mathbf{1}$: empty word)

Definition

The **quasi-shuffle product** $*_\diamond$ on $\mathbb{Q}\langle L\rangle$ is defined as the $\mathbb{Q}$-bilinear product satisfying $\mathbf{1} *_\diamond w = w *_\diamond \mathbf{1} = w$ for any word $w \in \mathbb{Q}\langle L\rangle$ and

$$aw *_\diamond bv = a(w *_\diamond bv) + b(aw *_\diamond v) + (a \diamond b)(w *_\diamond v)$$

for any letters $a, b \in L$ and words $w, v \in \mathbb{Q}\langle L\rangle$.

Theorem (Hoffman)

$(\mathbb{Q}\langle L\rangle, *_\diamond)$ *is a commutative* $\mathbb{Q}$*-algebra. Moreover, this algebra can be equipped with the structure of a Hopf algebra with the coproduct given by*

$$\Delta(w) = \sum_{uv=w} u \otimes v\,.$$

- **Harmonic product $*$**: $L_z = \{z_k \mid k \geq 1\}$ and $z_{k_1} \diamond z_{k_2} = z_{k_1+k_2}$ for all $k_1, k_2 \geq 1$.

$$z_2 * z_3 = z_2 z_3 + z_3 z_2 + z_5 \,.$$

(Compare with: $\zeta(2)\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$)

- **Shuffle product $ш$**: $L_{xy} = \{x, y\}$ and $a \diamond b = 0$ for $a, b \in L_{xy}$.

$$xy \,ш\, xxy = xyxxy + 3xxyxy + 6xxxyy \,.$$

By identifying $z_k \leftrightarrow \overbrace{x \cdots x}^{k-1} y$ we can also equip $\mathbb{Q}\langle L_z \rangle$ with the shuffle product, e.g.

$$z_2 \,ш\, z_3 = z_2 z_3 + 3 z_3 z_2 + 6 z_4 z_1 \,.$$

(Compare with: $\zeta(2)\zeta(3) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$)

- **Index shuffle product $\overline{ш}$**: $L_z = \{z_k \mid k \geq 1\}$ and $z_{k_1} \diamond z_{k_2} = 0$ for all $k_1, k_2 \geq 1$

$$z_2 \overline{ш} z_3 = z_2 z_3 + z_3 z_2 \,.$$

Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra.

Definition

1. A **mould** with values in $\mathcal{A}$ is a family $Z = (Z^{(r)})_{r\geq 0}$ with $Z^{(r)} \in \mathcal{A}[[X_1, \dots, X_r]]$.
2. For a mould Z with

$$Z^{(r)}(X_1, \dots, X_r) = \sum_{k_1,\dots,k_r\geq 1} z(k_1, \dots, k_r) X_1^{k_1-1} \dots X_r^{k_r-1}$$

we define its **coefficient map** as the $\mathbb{Q}$-linear map given by $\varphi_Z(\mathbf{1}) = Z^{(0)}$ and on the generators by

$$\begin{aligned} \varphi_Z : \mathbb{Q}\langle L_z\rangle &\longrightarrow \mathcal{A} \\ z_{k_1} \dots z_{k_r} &\longmapsto z(k_1, \dots, k_r)\,. \end{aligned}$$

Definition

1. A mould Z is called **$\diamond$-symmetril** if its coefficient map φ_Z gives an algebra homomorphism
$$\varphi_Z : (\mathbb{Q}\langle L_z\rangle, *_\diamond) \longrightarrow \mathcal{A}.$$
2. If $\diamond$ is given by $z_{k_1} \diamond z_{k_2} = z_{k_1+k_2}$ then we call a $\diamond$-symmetril mould **symmetril**. ($\leftrightsquigarrow$ harmonic product)
3. If $\diamond$ is given by $z_{k_1} \diamond z_{k_2} = 0$ then we call a $\diamond$-symmetril mould **symmetral**. ($\leftrightsquigarrow$ index shuffle product)

Example: The mould of **harmonic regularized multiple zeta values** $\mathfrak{z}$, whose depth r part is defined by
$$\mathfrak{z}(X_1,\dots,X_r) = \sum_{k_1,\dots,k_r\geq 1} \zeta^*(k_1,\dots,k_r) X_1^{k_1-1}\dots X_r^{k_r-1}.$$
is symmetril.

Let Z be a mould with $Z^{(1)}(X) = \sum_{k\geq 1} z(k)X^{k-1}$. Define the elements $\gamma_k^Z \in \mathcal{A}$ by

$$\sum_{k=0}^{\infty} \gamma_k^Z X^k := \exp\left(\sum_{n=2}^{\infty} \frac{(-1)^n}{n} z(n)X^n\right) .$$

With this we define the mould Z_γ by

$$Z_\gamma^{(r)}(X_1,\dots,X_r) = \sum_{j=0}^{r} \gamma_j^Z Z^{(r-j)}(X_1+\dots+X_{r-j},\dots,X_1+X_2,X_1).$$

Definition

We say a mould Z **satisfies the double shuffle relations** if Z is symmetril and Z_γ is symmetral.

Definition

We say a mould Z **satisfies the double shuffle relations** if Z is symmetril and Z_γ is symmetral.

In lowest depth, this means that if Z satisfies the double shuffle relations, then

$$
\begin{aligned}
Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}\,, \\
Z_\gamma(X_1)Z_\gamma(X_2) &= Z_\gamma(X_1, X_2) + Z_\gamma(X_2, X_1) \\
&= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + \gamma_2^Z\,.
\end{aligned}
$$

Theorem (Ecalle, Ihara-Kaneko-Zagier, Racinet, ...)

The mould of harmonic regularized multiple zeta values $\mathfrak{z}$ satisfies the double shuffle relations.

Theorem (Drinfeld + Furusho, Racinet)

There exists a mould $\mathfrak{b}$ with values in $\mathbb{Q}$, with the following properties.

1. *$\mathfrak{b}$ satisfies the double shuffle relations.*
2. *For all $r \geq 1$, $\mathfrak{b}(-X_1, \dots, -X_r) = (-1)^r \mathfrak{b}(X_1, \dots, X_r)$.*
3. *In depth one $\mathfrak{b}$ is given by*

$$\mathfrak{b}(X) = -\sum_{k \geq 2} \frac{B_k}{2k!} X^{k-1} = \sum_{m \geq 1} \frac{\zeta(2m)}{(2\pi i)^{2m}} X^{2m-1}\,.$$

This mould is not unique, but in the following, we will fix one choice of such a mould $\mathfrak{b}$ with coefficients β, i.e.

$$\mathfrak{b}(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} \beta(k_1, \dots, k_r) X_1^{k_1 - 1} \dots X_r^{k_r - 1}\,.$$

	Numbers	Functions / q-series
“single” objects	Riemann zeta values $\zeta(k) = \sum_{m>0} \frac{1}{m^k}$	Eisenstein series $\mathbb{G}_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n$ $(q = e^{2\pi i \tau})$
“multiple” objects	Multiple zeta values $\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}$	Multiple Eisenstein series $\mathbb{G}_{k_1,\ldots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}} = \zeta(k_1, \ldots, k_r) + \sum_{n>0} a_n q^n$
relations object point of view	Double shuffle relations $\zeta(2) \cdot \zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$ $= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$ $\zeta(2)^2 = \frac{5}{2}\zeta(4)$	$G_k = (-2\pi i)^{-k} \mathbb{G}_k$ $G_2^2 = \frac{5}{2} G_4 - \frac{1}{2} q \frac{d}{dq} G_2$
Mould / generating series point of view	Symmetril & Symmetral $Z(X_1)Z(X_2) = Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}\,,$ $Z_\gamma(X_1)Z_\gamma(X_2) = Z_\gamma(X_1, X_2) + Z_\gamma(X_2, X_1)\,.$	Symmetril & Swap invariant

Let $\mathcal{A}$ be a $\mathbb{Q}$-algebra, define $L_z^{\mathrm{bi}} = \{z_d^k \mid k \geq 1, d \geq 0\}$ and write $* = *_\diamond$ for $z_{d_1}^{k_1} \diamond z_{d_2}^{k_2} = z_{d_1+d_2}^{k_1+k_2}$.

Definition

1. A **bimould** with values in $\mathcal{A}$ is a family $B = (B^{(r)})_{r\geq 0}$ with $B^{(r)} \in \mathcal{A}[[X_1, \ldots, X_r, Y_1, \ldots, Y_r]]$.
2. For a bimould B with

$$B\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} = \sum_{\substack{k_1,\ldots,k_r \geq 1 \\ d_1,\ldots,d_r \geq 0}} b\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix} X_1^{k_1-1} \cdots X_r^{k_r-1} \frac{Y_1^{d_1}}{d_1!} \cdots \frac{Y_1^{d_r}}{d_r!}$$

we define its **coefficient map** as the $\mathbb{Q}$-linear map given by $\varphi_B(\mathbf{1}) = B^{(0)}$ and on the generators by

$$\begin{aligned} \varphi_B : \mathbb{Q}\langle L_z^{\mathrm{bi}} \rangle &\longrightarrow \mathcal{A} \\ z_{d_1}^{k_1} \cdots z_{d_r}^{k_r} &\longmapsto b\begin{pmatrix} k_1, \ldots, k_r \\ d_1, \ldots, d_r \end{pmatrix}. \end{aligned}$$

Definition

1. A bimould B is called **$\diamond$-symmetril** if its coefficient map φ_B gives an algebra homomorphism

$$\varphi_B : (\mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle, *_\diamond) \longrightarrow \mathcal{A}.$$

2. If $\diamond$ is given by $z_{d_1}^{k_1} \diamond z_{d_2}^{k_2} = z_{d_1+d_2}^{k_1+k_2}$ then we call a $\diamond$-symmetril bimould **symmetril**.

If B is symmetril then it satisfies in lowest depth

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2},$$

which is similar to the relation satisfied by a symmetril mould Z

$$Z(X_1)Z(X_2) = Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}.$$

Let B and C two bimoulds with values in $\mathcal{A}$. The **mould product** $B \times C$ is the bimould given by

$$(B \times C)\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{j=0}^{r} B\begin{pmatrix} X_1, \dots, X_j \\ Y_1, \dots, Y_j \end{pmatrix} C\begin{pmatrix} X_{j+1}, \dots, X_r \\ Y_{j+1}, \dots, Y_r \end{pmatrix}.$$

Proposition

If B and C are $\diamond$-symmetril then $B \times C$ is $\diamond$-symmetril.

Proof: The coefficient map of $B \times C$ is the convolution product of φ_B and φ_C, i.e.

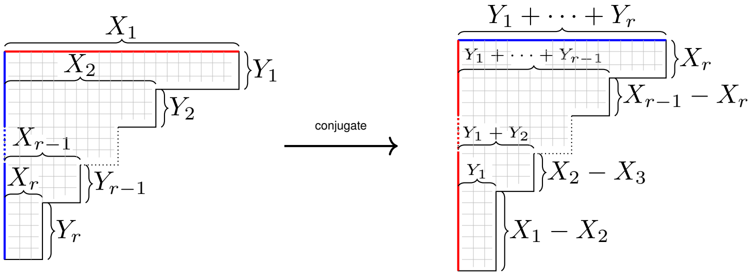
$$\varphi_{B\times C} = m \circ (\varphi_B \otimes \varphi_C) \circ \Delta\,,$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the multiplication on $\mathcal{A}$ and Δ is the deconcatination coproduct on $\mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle$. $\square$

Definition

A bimould B is called **swap invariant** if for all $r \geq 1$

$$B\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = B\begin{pmatrix} Y_1 + \dots + Y_r, Y_1 + \dots + Y_{r-1}, \dots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \dots, X_2 - X_3, X_1 - X_2 \end{pmatrix}$$



Example: If B is swap invariant we have $B\binom{X}{Y} = B\binom{Y}{X}$, which gives, for example, $b\binom{1}{1} = b\binom{2}{0}$.

Definition

For a mould Z, we define the bimould B^Z by

$$B^Z\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} = \sum_{j=0}^{r} Z_\gamma(Y_1, \ldots, Y_j) Z(X_{j+1}, \ldots, X_r)\,.$$

Recall that by definition

$$Z_\gamma^{(r)}(Y_1, \ldots, Y_r) = \sum_{j=0}^{r} \gamma_j^Z Z^{(r-j)}(Y_1 + \cdots + Y_{r-j}, \ldots, Y_1 + Y_2, Y_1)\,.$$

Proposition

- For any mould Z the bimould B^Z is swap invariant,
- If Z satisfies the double shuffle relations then B^Z is symmetril.

③ Algebraic setup - Swap invariant & symmetril bimould

Z satisfies the double shuffle relations $\Rightarrow B^Z$ is swap invariant & symmetril.

Question ("$\Leftarrow$" ?)

Does a swap invariant & symmetril bimould B give a mould Z which satisfies the double shuffle relations by setting

$$Z(X_1,\dots,X_r) = B\binom{X_1,\dots,X_r}{0,\dots,0}?$$

No, not in general: Let B swap invariant & symmetril bimould. Then one can show that its coefficient satisfy

$$b\binom{2}{0}^2 = \frac{5}{2}b\binom{4}{0} - b\binom{3}{1}.$$

Compare this to

$$G_2^2 = \frac{5}{2}G_4 - \frac{1}{2}q\frac{d}{dq}G_2, \qquad \text{and} \qquad \zeta(2)^2 = \frac{5}{2}\zeta(4).$$

$\longrightarrow$ The coefficients of an swap invariant & symmetril bimould "behave like Eisenstein series".

	Numbers	Functions / q-series
"single" objects	**Riemann zeta values** $$\zeta(k)=\sum_{m>0}\frac{1}{m^k}$$	**Eisenstein series** $$\mathbb{G}_k(\tau)=\zeta(k)+\frac{(-2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty}\sigma_{k-1}(n)q^n$$ $(q=e^{2\pi i\tau})$
"multiple" objects	**Multiple zeta values** $$\zeta(k_1,\dots,k_r)=\sum_{m_1>\cdots>m_r>0}\frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$$	**Multiple Eisenstein series** $$\mathbb{G}_{k_1,\dots,k_r}(\tau)=\sum_{\substack{\lambda_1\succ\cdots\succ\lambda_r\succ 0\\ \lambda_i\in\mathbb{Z}\tau+\mathbb{Z}}}\frac{1}{\lambda_1^{k_1}\cdots\lambda_r^{k_r}}=\zeta(k_1,\dots,k_r)+\sum_{n>0}a_nq^n$$
relations object point of view	**Double shuffle relations** $$\zeta(2)\cdot\zeta(3)=\zeta(2,3)+\zeta(3,2)+\zeta(5)$$ $$=\zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)$$ $$\zeta(2)^2=\frac{5}{2}\zeta(4)$$	$$G_2^2=\frac{5}{2}G_4-\frac{1}{2}q\frac{d}{dq}G_2$$ $$G_k=(-2\pi i)^{-k}\mathbb{G}_k$$ $$b\binom{2}{0}^2=\frac{5}{2}b\binom{4}{0}-b\binom{3}{1}$$
Mould / generating series point of view	**Symmetril & Symmetral** $$Z(X_1)Z(X_2)=Z(X_1,X_2)+Z(X_2,X_1)+\frac{Z(X_1)-Z(X_2)}{X_1-X_2},$$ $$Z_\gamma(X_1)Z_\gamma(X_2)=Z_\gamma(X_1,X_2)+Z_\gamma(X_2,X_1).$$	**Symmetril & Swap invariant** $$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2}=B\binom{X_1,X_2}{Y_1,Y_2}+B\binom{X_2,X_1}{Y_2,Y_1}+\frac{B\binom{X_1}{Y_1+Y_2}-B\binom{X_2}{Y_1+Y_2}}{X_1-X_2}$$ $$B\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}=B\binom{Y_1+\cdots+Y_r,Y_1+\cdots+Y_{r-1},\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_2-X_3,X_1-X_2}$$

	Numbers	Functions / q-series
Family of relations	Double shuffle relations Symmetril & Symmetral $Z(X_1)Z(X_2) = Z(X_1,X_2) + Z(X_2,X_1) + \frac{Z(X_1)-Z(X_2)}{X_1-X_2}\,,$ $Z_\gamma(X_1)Z_\gamma(X_2) = Z_\gamma(X_1,X_2) + Z_\gamma(X_2,X_1)\,.$	Symmetril & Swap invariant $B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1,X_2}{Y_1,Y_2} + B\binom{X_2,X_1}{Y_2,Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1-X_2}$ $B\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = B\binom{Y_1+\dots+Y_r, Y_1+\dots+Y_{r-1},\dots,Y_1+Y_2,Y_1}{X_r, X_{r-1}-X_r,\dots,X_2-X_3,X_1-X_2}$
Formal objects	Formal multiple zeta values	Formal (bi-)multiple Eisenstein series
Rational / combinatorial realization	Rational solution to double shuffle equations $\beta(k) = -\frac{B_k}{2k!}\quad \beta(k_1,\dots,k_r) \in \mathbb{Q}$	Combinatorial (bi-)multiple Eisenstein series
Real / analytic realization	(harmonic regularized) Multiple zeta values $\zeta(k_1,\dots,k_r) = \sum_{m_1>\dots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$	(??? extension) Multiple Eisenstein series $\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1\succ\dots\succ\lambda_r\succ 0\\ \lambda_i\in\mathbb{Z}\tau+\mathbb{Z}}} \frac{1}{\lambda_1^{k_1}\cdots\lambda_r^{k_r}} = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_n q^n$

(Rough) Let S be the ideal in $(\mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle, *)$ generated by the "swap invariance relations", e.g. $z_1^1 - z_0^2 \in S$.

Definition

The algebra of **formal multiple Eisenstein series** is defined by

$$\mathcal{G}^{\mathfrak{f}} = {}^{\mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle}\!\big/_{S}$$

and we denote the class of a word $z_{d_1}^{k_1} \dots z_{d_r}^{k_r}$ by $G_{\mathfrak{f}}\binom{k_1,\dots,k_r}{d_1,\dots,d_r}$ and set $G_{\mathfrak{f}}(k_1,\dots,k_r) := G_{\mathfrak{f}}\binom{k_1,\dots,k_r}{0,\dots,0}$.

Theorem (B.-Matthes-van-Ittersum (2022+))

The following map gives a derivation on $\mathcal{G}^{\mathfrak{f}}$

$$\partial G_{\mathfrak{f}}\begin{pmatrix} k_1,\dots,k_r \\ d_1,\dots,d_r \end{pmatrix} = \sum_{j=1}^{r} k_j G_{\mathfrak{f}}\begin{pmatrix} k_1,\dots,k_j+1,\dots,k_r \\ d_1,\dots,d_j+1,\dots,d_r \end{pmatrix}.$$

As an analogue of $G_2^2 = \frac{5}{2}G_4 - \frac{1}{2}q\frac{d}{dq}G_2$ we get $G_{\mathfrak{f}}(2)^2 = \frac{5}{2}G_{\mathfrak{f}}(4) - \frac{1}{2}\partial G_{\mathfrak{f}}(2)$.

④ Formal MES - Formal multiple Eisenstein series

Theorem (B.-Matthes-van-Ittersum (2022+))

1. *The space of formal modular forms* $\mathcal{M}^{\mathfrak{f}} = \mathbb{Q}[G_{\mathfrak{f}}(4), G_{\mathfrak{f}}(6)]$ *is isomorphic to the space of modular forms.*
2. *The space of formal quasi-modular forms* $\widetilde{\mathcal{M}}^{\mathfrak{f}} = \mathbb{Q}[G_{\mathfrak{f}}(2), G_{\mathfrak{f}}(4), G_{\mathfrak{f}}(6)]$ *is isomorphic to the space of quasi-modular forms as differential algebras.*
3. *There exist an ideal* N*, such that the algebra* $\mathcal{Z}^{\mathfrak{f}} = {}^{\mathcal{G}^{\mathfrak{f}}}\!/_{N}$ *is isomorphic to the algebra of* **formal multiple zeta values** *(defined by Racinet).*

Conjecture ($\mathfrak{sl}_2$-action)

There exist a unique derivation $\mathfrak{d}$ on $\mathcal{G}^{\mathfrak{f}}$ such that the triple $(\partial, W, \mathfrak{d})$ is an $\mathfrak{sl}_2$-triple, i.e.

$$[W, \partial] = 2\partial, \quad [W, \mathfrak{d}] = -2\mathfrak{d}, \qquad [\mathfrak{d}, \partial] = W\,,$$

where W is the weight operator.

We have an explicit conjectured construction of the derivation $\mathfrak{d}$. This $\mathfrak{sl}_2$-action would generalize the classical $\mathfrak{sl}_2$-action on the space of quasi-modular forms.

	Numbers	Functions / q-series
Family of relations	Double shuffle relations Symmetril & Symmetral $Z(X_1)Z(X_2) = Z(X_1,X_2) + Z(X_2,X_1) + \frac{Z(X_1)-Z(X_2)}{X_1-X_2}\,,$ $Z_\gamma(X_1)Z_\gamma(X_2) = Z_\gamma(X_1,X_2) + Z_\gamma(X_2,X_1)\,.$	Symmetril & Swap invariant $B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1,X_2}{Y_1,Y_2} + B\binom{X_2,X_1}{Y_2,Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1-X_2}$ $B\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = B\binom{Y_1+\dots+Y_r, Y_1+\dots+Y_{r-1},\dots,Y_1+Y_2,Y_1}{X_r, X_{r-1}-X_r,\dots,X_2-X_3,X_1-X_2}$
Formal objects	Formal multiple zeta values $\mathcal{Z}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}} / N$	Formal (bi-)multiple Eisenstein series $\mathcal{G}^{\mathfrak{f}} = \mathbb{Q}\langle L_z^{\mathrm{bi}}\rangle / S$ $\quad G_{\mathfrak{f}}\binom{2}{0}^2 = \frac{5}{2}G_{\mathfrak{f}}\binom{4}{0} - G_{\mathfrak{f}}\binom{3}{1}$
Rational / combinatorial realization	Rational solution to double shuffle equations $\beta(k) = -\frac{B_k}{2k!}$ $\quad \beta(k_1,\dots,k_r) \in \mathbb{Q}$	Combinatorial (bi-)multiple Eisenstein series
Real / analytic realization	(harmonic regularized) Multiple zeta values $\zeta(k_1,\dots,k_r) = \sum_{m_1>\dots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$	(??? extension) Multiple Eisenstein series $\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1\succ\dots\succ\lambda_r\succ 0\\ \lambda_i\in\mathbb{Z}\tau+\mathbb{Z}}} \frac{1}{\lambda_1^{k_1}\cdots\lambda_r^{k_r}} = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_n q^n$

Theorem ((work in progress) B.-Burmester (2022+))

There exist a swap invariant & symmetril bimould $\mathfrak{G}$ *with values in* $\mathbb{Q}[[q]]$

$$\mathfrak{G}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \sum_{\substack{k_1,\dots,k_r\geq 1\\ d_1,\dots,d_r\geq 0}} G\binom{k_1,\dots,k_r}{d_1,\dots,d_r} X_1^{k_1-1}\dots X_r^{k_r-1}\frac{Y_1^{d_1}}{d_1!}\dots\frac{Y_1^{d_r}}{d_r!}$$

such that the coefficients in depth one are given by Eisenstein series and their derivatives $(k > d \geq 0)$

$$G\binom{k}{d} = \frac{(k-d-1)!}{(k-1)!}\left(q\frac{d}{dq}\right)^d G_{k-d}\,.$$

Define the **combinatorial multiple Eisenstein series** for $k_1,\dots,k_r \geq 1$ by

$$G(k_1,\dots,k_r) := G\binom{k_1,\dots,k_r}{0,\dots,0}\,.$$

Denote the space spanned by all combinatorial multiple Eisenstein by

$$\mathcal{G} = \mathbb{Q} + \big\langle G(k_1,\dots,k_r) \mid r \geq 1, k_1,\dots,k_r \geq 1 \big\rangle_{\mathbb{Q}} \subset \mathbb{Q}[[q]]\,.$$

Theorem (B.-Burmester (2022+))

1. *The space $\mathcal{G}$ is a $\mathbb{Q}$-algebra which contains the space of (quasi-)modular forms with rational coefficients.*
2. *The combinatorial multiple Eisenstein series give an algebra homomorphism*

$$\begin{aligned} G : (\mathbb{Q}\langle L_z\rangle, *) &\longrightarrow \mathcal{G} \\ w = z_{k_1}\dots z_{k_r} &\longmapsto G(w) := G(k_1,\dots,k_r)\,. \end{aligned}$$

3. *$\mathcal{G}$ is closed under $q\frac{d}{dq}$ and for any $w \in \mathbb{Q}\langle L_z\rangle$ we have*

$$q\frac{d}{dq}G(w) = G(z_2 * w - z_2 ⧢ w)\,.$$

The combinatorial multiple Eisenstein series have the form

$$G(k_1,\dots,k_r) = \beta(k_1,\dots,k_r) + \text{products of } \beta \text{ and } g \text{ in lower depths} + g(k_1,\dots,k_r)\,.$$

Example: $$G(3,2) = \beta(3,2) + 3\beta(3)g(2) + 2\beta(2)g(3) + g(3,2)$$

Therefore they can be seen as an interpolation between the harmonic regularized multiple zeta values and the rational solutions to double shuffle equations: For all $k_1,\dots,k_r \geq 1$ we have

$$\lim_{q\to 1}^{*} (1-q)^{k_1+\dots+k_r} G(k_1,\dots,k_r) = \zeta^*(k_1,\dots,k_r)$$
$$\lim_{q\to 0} G(k_1,\dots,k_r) = \beta(k_1,\dots,k_r)\,.$$

Here $\lim^*_{q\to 1}$ means that for $k_1 = 1$ one needs to use a regularized limit (B.-van-Ittersum 2022+)

	Numbers	Functions / q-series
Family of relations	Double shuffle relations Symmetril & Symmetral $Z(X_1)Z(X_2) = Z(X_1,X_2) + Z(X_2,X_1) + \frac{Z(X_1)-Z(X_2)}{X_1-X_2}\,,$ $Z_\gamma(X_1)Z_\gamma(X_2) = Z_\gamma(X_1,X_2) + Z_\gamma(X_2,X_1)\,.$	Symmetril & Swap invariant $B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1,X_2}{Y_1,Y_2} + B\binom{X_2,X_1}{Y_2,Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1-X_2}$ $B\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = B\binom{Y_1+\dots+Y_r, Y_1+\dots+Y_{r-1},\dots,Y_1+Y_2,Y_1}{X_r, X_{r-1}-X_r,\dots,X_2-X_3,X_1-X_2}$
Formal objects	Formal multiple zeta values $\mathcal{Z}^{\mathfrak{f}} = \mathcal{G}^{\mathfrak{f}} / N$	Formal (bi-)multiple Eisenstein series $\mathcal{G}^{\mathfrak{f}} = \mathbb{Q}\langle L_z^{\mathrm{bi}} \rangle / S$ $\quad G_{\mathfrak{f}}\binom{2}{0}^2 = \frac{5}{2}G_{\mathfrak{f}}\binom{4}{0} - G_{\mathfrak{f}}\binom{3}{1}$
Rational / combinatorial realization	Rational solution to double shuffle equations $\beta(k) = -\frac{B_k}{2k!}$ $\quad \beta(k_1,\dots,k_r) \in \mathbb{Q}$	$\xleftarrow{q\to 0}$ Combinatorial (bi-)multiple Eisenstein series $G(k_1,\dots,k_r) \in \mathbb{Q}[[q]]$
Real / analytic realization	(harmonic regularized) Multiple zeta values $\zeta(k_1,\dots,k_r) = \sum_{m_1>\dots>m_r>0} \frac{1}{m_1^{k_1}\cdots m_r^{k_r}}$	$\xleftarrow{q\to 1}$ (??? extension of) Multiple Eisenstein series $\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1\succ\dots\succ\lambda_r\succ 0\\ \lambda_i\in\mathbb{Z}\tau+\mathbb{Z}}} \frac{1}{\lambda_1^{k_1}\cdots\lambda_r^{k_r}} = \zeta(k_1,\dots,k_r) + \sum_{n>0} a_n q^n$

⑤ Combinatorial MES - Construction of the bimould $\mathfrak{G}$

With $L_m\binom{X}{Y} = \frac{e^{X+mY}q^m}{1-e^X q^m}$ define the bimould $\mathfrak{g}$ with values in $\mathbb{Q}[[q]]$ by

$$\mathfrak{g}\begin{pmatrix} X_1,\dots,X_r \\ Y_1,\dots,Y_r \end{pmatrix} = \sum_{m_1>\dots>m_r>0} L_{m_1}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \dots L_{m_r}\begin{pmatrix} X_r \\ Y_r \end{pmatrix}.$$

Theorem (B. 2013)

The bimould $\mathfrak{g}$ is swap invariant.

The coefficients generalize the q-series g. This bimould is not symmetril, but satisfies, for example,

$$\mathfrak{g}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}\mathfrak{g}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix} = \mathfrak{g}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} + \mathfrak{g}\begin{pmatrix} X_2, X_1 \\ Y_2, Y_1 \end{pmatrix} + \frac{\mathfrak{g}\binom{X_1}{Y_1+Y_1} - \mathfrak{g}\binom{X_2}{Y_1+Y_1}}{X_1 - X_2}$$
$$+ \left(2\mathfrak{b}(X_2 - X_1) - \frac{1}{2}\right)\mathfrak{g}\begin{pmatrix} X_1 \\ Y_1 + Y_1 \end{pmatrix} + \left(2\mathfrak{b}(X_1 - X_2) - \frac{1}{2}\right)\mathfrak{g}\begin{pmatrix} X_2 \\ Y_1 + Y_1 \end{pmatrix}.$$

Using the swap invariance of $\mathfrak{g}$, the above relationship between $\mathfrak{g}$ and $\mathfrak{b}$ and the fact that $\mathfrak{b}$ satisfies the double shuffle relation, one can given an explicit (but complicated) construction of $\mathfrak{G}$.

Calculation of the Fourier expansion of multiple Eisenstein series

Multiple Eisenstein series

$$\mathbb{G}_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}}$$

$$\mathbb{G}_{k_1,k_2}(\tau) = \zeta(k_1,k_2) + \sum_{m>0} \Psi_{k_1}(m\tau)\zeta(k_2) + \sum_{m>0} \Psi_{k_1,k_2}(m\tau) + \sum_{m_1>m_2>0} \Psi_{k_1}(m_1\tau)\Psi_{k_2}(m_2\tau)$$

$$\Psi_{k_1,\dots,k_r}(\tau) := \sum_{\substack{n_1 > \cdots > n_r \\ n_i \in \mathbb{Z}}} \frac{1}{(\tau+n_1)^{k_1} \cdots (\tau+n_r)^{k_r}}.$$

mould product decomposition

MZV

$$\zeta(k_1,\dots,k_r)$$

Sums of multitangent functions

$$\sum_{m>0} \Psi_{k_1,k_2}(m\tau) + \sum_{m_1>m_2>0} \Psi_{k_1}(m_1\tau)\Psi_{k_2}(m_2\tau)$$

Reduction multitangent to monotangent

$$\Psi_{k_1,\dots,k_r}(\tau) = \sum_{j=2}^{K} \alpha_{K-j}\Psi_j(\tau)$$

q-MZV (sums of monotangent functions)

$$g(k_1,\dots,k_r) = (-2\pi i)^{-(k_1+\cdots+k_r)} \sum_{m_1>\cdots>m_r>0} \Psi_{k_1}(m_1\tau)\dots\Psi_{k_r}(m_r\tau)$$

Construction of combinatorial multiple Eisenstein series

Symmetril & swap invariant bimould

$$G\binom{k_1,\dots,k_r}{d_1,\dots,d_r} \qquad \mathfrak{G}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$$

Mould product

$$\mathfrak{G} = \mathfrak{g}^* \times \mathfrak{b}$$

Symmetril & swap invariant bimould

$$\mathfrak{b}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$$

B^Z

Rational solution for double shuffle equations

$$\mathfrak{b}(X_1,\dots,X_r)$$

Sums of multiple version of L: Symmetril bimould $\mathfrak{g}^*$

$$\mathfrak{g}^*\binom{X_1,X_2}{Y_1,Y_2} = \sum_{m_1>m_2>0} \mathfrak{L}_{m_1}\binom{X_1}{Y_1}\mathfrak{L}_{m_2}\binom{X_2}{Y_2} + \sum_{m>0}\mathfrak{L}_m\binom{X_1,X_2}{Y_1,Y_2}$$

Symmetril bimould

$$\mathfrak{L}_m$$

Define multiple version of L by b and single version of L

$$\mathfrak{L}_m\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \sum_{j=1}^{r}\mathfrak{b}\binom{X_1-X_j,\dots,X_{j-1}-X_j}{Y_1,\dots,Y_{j-1}} L_m\binom{X_j}{Y_1+\cdots+Y_r}\tilde{\mathfrak{b}}\binom{X_r-X_j,\dots,X_{j+1}-X_j}{Y_r,\dots,Y_{j+1}}$$

The series g (sums of single version of L)

$$\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \sum_{m_1>\cdots>m_r>0} L_{m_1}\binom{X_1}{Y_1}\dots L_{m_r}\binom{X_r}{Y_r}$$

Recall $L_m\binom{X}{Y} = \frac{e^{X+mY}q^m}{1-e^X q^m}$ and set $\tilde{\mathfrak{b}}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \sum_{i=0}^{r} \frac{(-1)^i}{2^i i!}\mathfrak{b}\binom{X_{i+1},\dots,X_r}{-Y_1,\dots,-Y_{r-i}}$.

Definition

For $m \geq 1$ we define the bimould $\mathfrak{L}_m$ by defining $\mathfrak{L}_m\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$ as

$$\sum_{j=1}^{r} \mathfrak{b}\begin{pmatrix} X_1 - X_j, \dots, X_{j-1} - X_j \\ Y_1, \dots, Y_{j-1} \end{pmatrix} L_m\begin{pmatrix} X_j \\ Y_1 + \dots + Y_r \end{pmatrix} \tilde{\mathfrak{b}}\begin{pmatrix} X_r - X_j, \dots, X_{j+1} - X_j \\ Y_r, \dots, Y_{j+1} \end{pmatrix}.$$

The $L_m\binom{X}{Y}$ can be seen as the generating series of the "(bi-)combinatorial version" of the monotangent function $\Psi_k^{\text{comb}}(\tau) = \frac{1}{(k-1)!}\sum_{d>0} d^{k-1}q^d$ (defined by the Lipschitz formula instead of nested sum), since

$$\sum_{k\geq 1} \Psi_k^{\text{comb}}(m\tau)X^{k-1} = \sum_{k\geq 1}\frac{1}{(k-1)!}\sum_{d>0} d^{k-1}q^{md}X^{k-1} = \sum_{d>0} e^{dX}q^{md} = \frac{e^X q^m}{1-e^X q^m} = L_m\begin{pmatrix} X \\ 0 \end{pmatrix}.$$

The $\mathfrak{L}_m$ can then be seen as the generating series of (bi-)combinatorial version of the multitangent functions.

Lemma

Let B_m be a family of bimoulds which are $\diamond$-symmetril for all $m \geq 1$. Then the bimould C_M defined by

$$C_M\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} = \sum_{\substack{1 \leq j \leq r \\ 0=r_0<r_1<\cdots<r_{j-1}<r_j=r \\ M>m_1>\cdots>m_j>0}} \prod_{i=1}^{j} B_{m_i}\begin{pmatrix} X_{r_{i-1}+1}, \ldots, X_{r_i} \\ Y_{r_{i-1}+1}, \ldots, Y_{r_i} \end{pmatrix}$$

is $\diamond$-symmetril for all $M \geq 1$. **Proof:** *Show $C_{M+1} = B_M \times C_M$ and do induction on M.*

Definition

We define the bimould $\mathfrak{g}^*$ by

$$\mathfrak{g}^*\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} = \sum_{\substack{1 \leq j \leq r \\ 0=r_0<r_1<\cdots<r_{j-1}<r_j=r \\ m_1>\cdots>m_j>0}} \prod_{i=1}^{j} \mathfrak{L}_{m_i}\begin{pmatrix} X_{r_{i-1}+1}, \ldots, X_{r_i} \\ Y_{r_{i-1}+1}, \ldots, Y_{r_i} \end{pmatrix}.$$

Lemma $\implies$ if the $\mathfrak{L}_m$ are symmetril for all m then $\mathfrak{g}^*$ is symmetril.

Definition (B.-Burmester (2022+))

The bimould of combinatorial (bi)-multiple Eisenstein series is defined by $\mathfrak{G} = \mathfrak{g}^* \times \mathfrak{b}$.

Definition (B.-Burmester (2022+))

For $j \geq 0$ we define the bimould $\mathfrak{G}_j = (\mathfrak{G}_j^{(r)})_{r \geq 0}$ as follows. In the case $j = 0$ we set $\mathfrak{G}_0 = \mathfrak{b}$ and $\mathfrak{G}_j^{(r)} = 0$ for $r < j$. If $1 \leq j \leq r$ we define

$$\mathfrak{G}_j\begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = \sum_{\substack{0=r_0<r_1<\dots<r_j\leq r \\ m_1>\dots>m_j>0}} \prod_{i=1}^{j} \mathfrak{L}_{m_i}\begin{pmatrix} X_{r_{i-1}+1}, \dots, X_{r_i} \\ Y_{r_{i-1}+1}, \dots, Y_{r_i} \end{pmatrix} \mathfrak{b}\begin{pmatrix} X_{r_j+1}, \dots, X_r \\ Y_{r_j+1}, \dots, Y_r \end{pmatrix}.$$

Theorem (B.-Burmester (2022+))

The bimould $\mathfrak{G}_j$ is swap invariant for any $j \geq 0$ and we have $\mathfrak{G} = \sum_{j=0}^{r} \mathfrak{G}_j$, i.e. $\mathfrak{G}$ is swap invariant.

Let $\mathfrak{b} = B^{\mathfrak{b}}$ denote the bimould coming from the mould $\mathfrak{b}$, which satisfies the double shuffle relation. (i.e. the bimould $\mathfrak{b}$ is symmetril and swap invariant)

Example: In depth one and two the bimould $\mathfrak{G}$ is given by

$$\mathfrak{G}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} = \mathfrak{b}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} + \mathfrak{g}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix},$$

$$\begin{aligned}\mathfrak{G}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} &= \mathfrak{b}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} - \mathfrak{b}\begin{pmatrix} X_1 - X_2 \\ Y_2 \end{pmatrix}\mathfrak{g}\begin{pmatrix} X_1 \\ Y_1 + Y_2 \end{pmatrix} - \frac{1}{2}\mathfrak{g}\begin{pmatrix} X_1 \\ Y_1 + Y_2 \end{pmatrix} \\ &\quad + \mathfrak{b}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}\mathfrak{g}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} + \mathfrak{b}\begin{pmatrix} X_1 - X_2 \\ Y_1 \end{pmatrix}\mathfrak{g}\begin{pmatrix} X_2 \\ Y_1 + Y_2 \end{pmatrix} + \mathfrak{g}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix}.\end{aligned}$$

Open questions & future directions

There are still various open questions and possible research directions which are also suitable for bachelor, master & PhD projects.

1. Higher level analogues (cf. Kaneko-Tasaka 2013, Yuan-Zhao 2016).
2. Analytic realization of the formal multiple Eisenstein series.
3. Consider additional structures from modular forms, e.g. Hecke operators.
4. Extension of the Kronecker realization (B.-Kühn-Matthes 2021) to higher depths.
5. Connection to the Goncharov coproduct (cf. B.-Tasaka 2017).
6. Possible definition of q-Associators.
7. Basis & Dimension formulas (cf. B.-Kühn 2020).
8. Interpretation of the Broadhurst-Kreimer conjecture & exotic relations in this setup.
9. Adaptation of this setup for finite multiple zeta values (cf. Kaneko-Zagier, B.-Tasaka-Takeyama 2018).
10. etc.

Thank you for your attention.

Bonus - Connection to Goncharov coproduct

On the $\mathbb{Q}$-algebra $(\mathbb{Q}\langle L_z\rangle, ⧢)$ one can define the **Goncharov coproduct** Δ_G, which gives $(\mathbb{Q}\langle L_z\rangle, ⧢)$ the structure of a Hopf algebra.

There exist explicit formulas for Δ_G, e.g.

$$\Delta_G(z_3 z_2) = z_3 z_2 \otimes 1 + 3 z_3 \otimes z_2 + 2 z_2 \otimes z_3 + 1 \otimes z_3 z_2 \,.$$

Compare this to the Fourier expansion of $\mathbb{G}_{3,2}$:

$$\mathbb{G}_{3,2} = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + \underbrace{(-2\pi i)^5 g(3,2)}_{\tilde{g}(3,2):=} \,.$$

Theorem (B.-Tasaka 2017)

For $k_1, \dots, k_r \geq 2$ we have $\mathbb{G}_{k_1,\dots,k_r} = (m \circ (\zeta \otimes \tilde{g}) \circ \Delta_G)(z_{k_1} \dots z_{k_r})$.

We also have

$$G(3,2) = \beta(3,2) + 3\beta(3) g(2) + 2\beta(2) g(3) + g(3,2) = g(3,2) - \frac{1}{12} g(3) \,,$$

and by construction an analogue of the above theorem for combinatorial multiple Eisenstein series is expected.

Bonus - Modular forms - Definition

Complex upper half plane: $\mathbb{H} = \left\{x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}\,, y > 0\right\}$.

Definition

A holomorphic function $f \in \mathcal{O}(\mathbb{H})$ is called a **modular form of weight** $k \in \mathbb{Z}$ if it satisfies

- $f(\tau + 1) = f(\tau)\,,$
- $f(-\frac{1}{\tau}) = \tau^k f(\tau)\,,$

for all $\tau \in \mathbb{H}$ and if it has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n\,. \qquad (a_n \in \mathbb{C}, q = e^{2\pi i \tau})$$

- $\mathcal{M}_k$: space of all modular forms of weight k, $\mathcal{M} = \oplus_{k\geq 0}\mathcal{M}_k$ space of all modular forms.
- The space of **cusp forms** of weight k is defined by

$$\mathcal{S}_k = \left\{f \in \mathcal{M}_k \mid f = \sum_{\mathbf{n=1}}^{\infty} a_n q^n\right\} = \ker\big(\text{projection to const. term}\big)\,.$$

Bonus - Broadhurst-Kreimer conjecture

$\mathrm{gr}_r^{\mathrm{D}} \mathcal{Z}_k$: MZV of weight k and depth r modulo lower depths MZV.

Conjecture (Broadhurst-Kreimer, 1997)

The generating series of the dimensions of the weight- and depth-graded parts of multiple zeta values is given by

$$\sum_{k,r\geq 0} \dim_{\mathbb{Q}} \left(\mathrm{gr}_r^{\mathrm{D}} \mathcal{Z}_k\right) X^k Y^r = \frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4},$$

where

$$\mathsf{E}(X) = \frac{X^2}{1-X^2}, \quad \mathsf{O}(X) = \frac{X^3}{1-X^2}, \quad \mathsf{S}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)} = \sum_{k\geq 0} \dim \mathcal{S}_k X^k.$$

Observe that

$$\frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4}$$
$$= 1 + \left(\mathsf{E}(X)+\mathsf{O}(X)\right)Y + \Big(\big(\mathsf{E}(X)+\mathsf{O}(X)\big)\,\mathsf{O}(X) - \mathsf{S}(X)\Big)\,Y^2 + \cdots .$$

Bonus - Analogue for the double shuffle relation in small depth

As a consequence of the swap invariance the formal (and therefore also the combinatorial) bi-multiple Eisenstein series satisfy for $k_1, k_2 \geq 1, d_1, d_2 \geq 0$

$$G_{\mathfrak{f}}\binom{k_1}{d_1} G_{\mathfrak{f}}\binom{k_2}{d_2} = G_{\mathfrak{f}}\binom{k_1, k_2}{d_1, d_2} + G_{\mathfrak{f}}\binom{k_2, k_1}{d_2, d_1} + G_{\mathfrak{f}}\binom{k_1+k_2}{d_1+d_2}$$

$$= \sum_{\substack{l_1+l_2=k_1+k_2 \\ e_1+e_2=d_1+d_2 \\ l_1,l_2\geq 1, e_1,e_2\geq 0}} \left(\binom{l_1-1}{k_1-1}\binom{d_1}{e_1}(-1)^{d_1-e_1} + \binom{l_1-1}{k_2-1}\binom{d_2}{e_1}(-1)^{d_2-e_1} \right) G_{\mathfrak{f}}\binom{l_1, l_2}{e_1, e_2}$$

$$+ \frac{d_1! d_2!}{(d_1+d_2+1)!}\binom{k_1+k_2-2}{k_1-1} G_{\mathfrak{f}}\binom{k_1+k_2-1}{d_1+d_2+1}.$$

Example The $k_1 = 2, k_2 = 3, d_1 = d_2 = 0$ case gives

$$\begin{aligned} G_{\mathfrak{f}}(2)G_{\mathfrak{f}}(3) &= G_{\mathfrak{f}}(2,3) + G_{\mathfrak{f}}(3,2) + G_{\mathfrak{f}}(5) \\ &= G_{\mathfrak{f}}(2,3) + 3G_{\mathfrak{f}}(3,2) + 6G_{\mathfrak{f}}(4,1) + \partial G_{\mathfrak{f}}(3). \end{aligned}$$

Compare this to $\zeta(2)\cdot\zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5) = \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$.