A variant of the double shuffle relations and quasi modular forms

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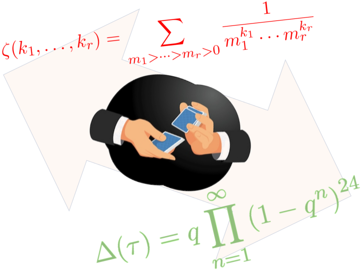


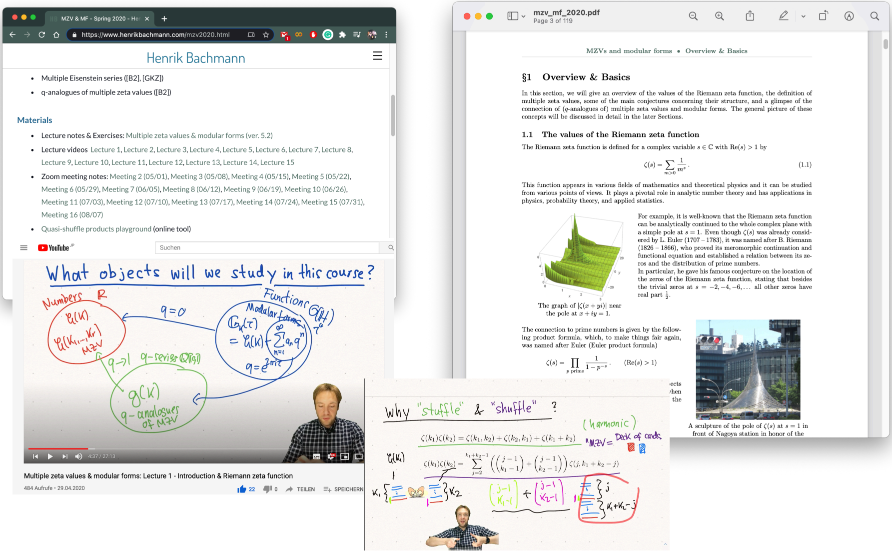
j.w. Annika Burmester Ulf Kühn & Nils Matthes

ACPMS Seminar, 4th December 2020

www.henrikbachmann.com

Lecture with notes & videos related to this talk

https://www.henrikbachmann.com/mzv2020.html



	Numbers	Functions / q-series
“single” objects		
“multiple” objects		
relations		

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object point of view	$\zeta(2)\cdot\zeta(3) = \zeta(2,3)+\zeta(3,2)+\zeta(5)$ $= \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)$	
generating series point of view	$Z(X_1)Z(X_2) = Z(X_1,X_2)+Z(X_2,X_1)+\frac{Z(X_1)-Z(X_2)}{X_1-X_2}$ $= Z(X_1+X_2,X_1)+Z(X_1+X_2,X_2)+z_2\,.$	

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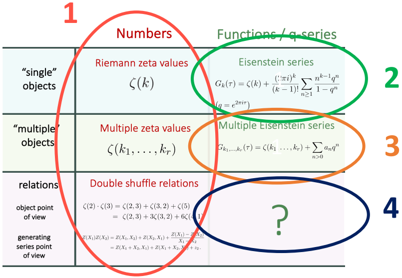
Plan of this talk



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1
Multiple zeta values & Double shuffle relations

2
Modular forms

3
Multiple Eisenstein series

4
"Double shuffle relations for functions"

Definition

For $k_1 \geq 2, k_2, \ldots, k_r \geq 1$ define the **multiple zeta value** (MZV)

$$\zeta(k_1, \ldots, k_r) = \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \in \mathbb{R}\,.$$

By r we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**.

- $\mathcal{Z}$: $\mathbb{Q}$-algebra of MZVs
- $\mathcal{Z}_k$: $\mathbb{Q}$-vector space of MZVs of weight k.

MZVs can also be written as **iterated integrals**, e.g.

$$\zeta(2,3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}\,.$$

There are two different ways to express the product of MZV in terms of MZV.

Harmonic product (coming from the definition as iterated sums)

Example in depth two ($k_1, k_2 \geq 2$)

$$\begin{aligned}\zeta(k_1)\cdot\zeta(k_2) &= \sum_{m>0}\frac{1}{m^{k_1}}\sum_{n>0}\frac{1}{n^{k_2}}\\ &= \sum_{m>n>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{n>m>0}\frac{1}{m^{k_1}n^{k_2}} + \sum_{m=n>0}\frac{1}{m^{k_1+k_2}}\\ &= \zeta(k_1,k_2)+\zeta(k_2,k_1)+\zeta(k_1+k_2)\,.\end{aligned}$$

Shuffle product (coming from the expression as iterated integrals)

Example in depth two ($k_1, k_2 \geq 2$)

$$\zeta(k_1)\cdot\zeta(k_2) = \sum_{j=2}^{k_1+k_2-1}\left(\binom{j-1}{k_1-1}+\binom{j-1}{k_2-1}\right)\zeta(j,k_1+k_2-j)\,.$$

These two product expressions give various $\mathbb{Q}$-linear relations between MZV.

Example

$$\begin{aligned}\zeta(2)\cdot\zeta(3) &\overset{\text{harmonic}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\\ &\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)\,.\end{aligned}$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0}\frac{1}{m^2n} = \zeta(2,1) = \zeta(3) = \sum_{m>0}\frac{1}{m^3}.$$

These follow from regularizing the double shuffle relations

$\rightsquigarrow$**extended double shuffle relations**.

Conjecture

All relations among MZVs are consequences of the extended double shuffle relations.

Conjecture

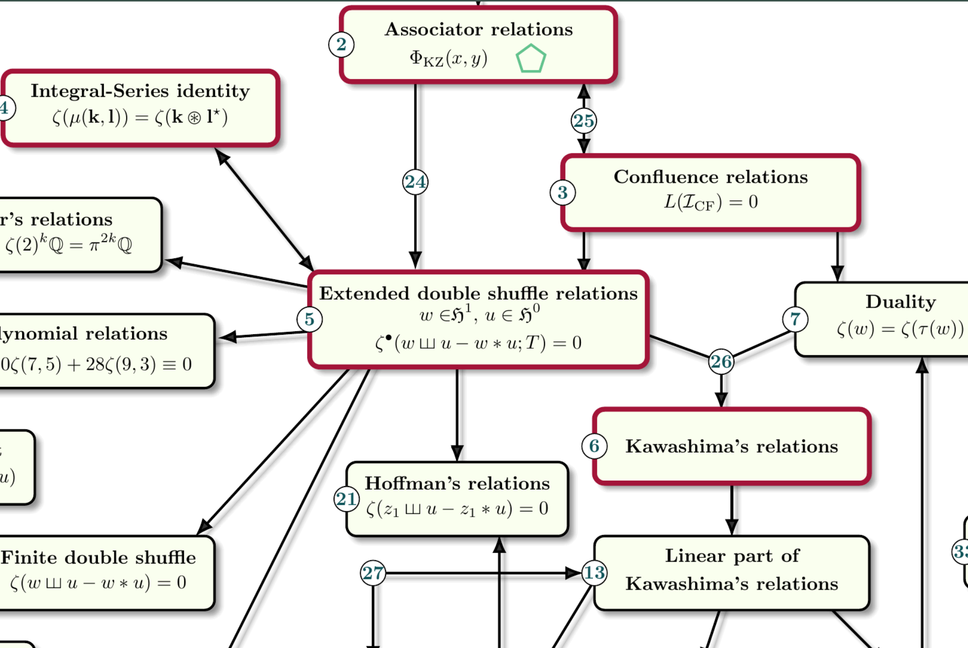
The space $\mathcal{Z}$ is graded by weight, i.e.

$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k \,.$$

- There are various different families of relations which conjecturally give all relations among MZV.
- Not for all of them it is known if they are equivalent to the extended double shuffle relations.

Overview of relations among MZV

For details see:
B. “Multiple zeta values & modular forms”, Lecture notes



Define the numbers $d_k \in \mathbb{Z}_{\geq 0}$ by

$$\sum_{k\geq 0} d_k X^k = \frac{1}{1 - X^2 - X^3}.$$

Conjecture (Zagier, 1994)

We have $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$ for all $k \geq 0$.

weight k	0	1	2	3	4	5	6	7	8	9	10	11	12
# of adm. ind.	1	0	1	2	4	8	16	32	64	128	256	512	1024
# of relations $\stackrel{?}{=}$	0	0	0	1	3	6	14	29	60	123	249	503	1012
d_k	1	0	1	1	1	2	2	3	4	5	7	9	12

Theorem (Terasoma (2002), Deligne–Goncharov (2005))

For all $k \geq 0$ we have $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$.

② Modular forms - Definition

Complex upper half plane: $\mathbb{H} = \left\{ x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}\, , y > 0 \right\}$.

Definition

A holomorphic function $f \in \mathcal{O}(\mathbb{H})$ is called a **modular form of weight** $k \in \mathbb{Z}$ if it satisfies

- $f(\tau + 1) = f(\tau)\,,$
- $f(-\frac{1}{\tau}) = \tau^k f(\tau)\,,$

for all $\tau \in \mathbb{H}$ and if it has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n \,. \qquad (a_n \in \mathbb{C}, q = e^{2\pi i \tau})$$

- $\mathcal{M}_k$: space of all modular forms of weight k.
- The space of **cusp forms** of weight k is defined by

$$\mathcal{S}_k = \{ f \in \mathcal{M}_k \mid f = \sum_{\mathbf{n=1}}^{\infty} a_n q^n \}\,.$$

For even $k \geq 4$ the **Eisenstein series** are defined by

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k} .$$

These have a Fourier expansion of the form

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n ,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ is the divisor sum.

Proposition

For every even $k \geq 4$ we have $G_k \in \mathcal{M}_k$ and

$$\mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}_k = \mathbb{C}[G_4, G_6] .$$

Are derivatives of modular forms again modular forms?... No

Define the **Eisenstein series of weight two** by

$$G_2(\tau) = \zeta(2) + (-2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n \,,$$

and the space of **quasi-modular forms** by

$$\widetilde{\mathcal{M}} = \mathbb{C}[G_2, G_4, G_6] \,.$$

Proposition

The space $\widetilde{\mathcal{M}}$ is the "smallest" ring with the following properties:

- It contains the ring of modular forms $\mathcal{M}$.
- It it closed under $\frac{d}{d\tau}$.

The first non-trivial cusp form is the **discriminant function** Δ

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots ,$$

which is a cusp form of weight 12.

Theorem

- For $k \geq 0$ the map $\mathcal{M}_k \to \mathcal{S}_{k+12}$ given by $f \mapsto \Delta \cdot f$ is an isomorphism of $\mathbb{C}$-vector spaces.
- The generating series for the dimension of cusp forms of weight k is given by

$$\mathsf{S}(X) = \sum_{k \geq 0} \dim_{\mathbb{C}} \mathcal{S}_k X^k = X^{12} \sum_{k \geq 0} \dim_{\mathbb{C}} \mathcal{M}_k X^k = \frac{X^{12}}{(1 - X^4)(1 - X^6)} .$$

$\mathrm{gr}_r^{\mathrm{D}} \mathcal{Z}_k$: MZV of weight k and depth r modulo lower depths MZV.

Conjecture (Broadhurst-Kreimer, 1997)

The generating series of the dimensions of the weight- and depth-graded parts of multiple zeta values is given by

$$\sum_{k,r\geq 0} \dim_{\mathbb{Q}} \left(\mathrm{gr}_r^{\mathrm{D}} \mathcal{Z}_k\right) X^k Y^r = \frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4},$$

where

$$\mathsf{E}(X) = \frac{X^2}{1-X^2}, \quad \mathsf{O}(X) = \frac{X^3}{1-X^2}, \quad \mathsf{S}(X) = \frac{X^{12}}{(1-X^4)(1-X^6)}.$$

Observe that

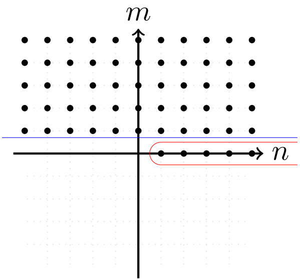
$$\frac{1+\mathsf{E}(X)Y}{1-\mathsf{O}(X)Y+\mathsf{S}(X)Y^2-\mathsf{S}(X)Y^4}$$
$$= 1 + \left(\mathsf{E}(X)+\mathsf{O}(X)\right)Y + \Big(\left(\mathsf{E}(X)+\mathsf{O}(X)\right)\mathsf{O}(X) - \mathsf{S}(X)\Big)Y^2 + \cdots .$$

Let $\tau \in \mathbb{H}$. We define an order $\succ$ on the lattice $\mathbb{Z}\tau + \mathbb{Z}$ by setting

$$\lambda_1 \succ \lambda_2 :\Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for $\lambda_1, \lambda_2 \in \mathbb{Z}\tau + \mathbb{Z}$ and the following set of positive lattice points

$$P := \{m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \mid m > 0 \vee (m = 0 \wedge n > 0)\} = U \cup R\,.$$



In other words: $\lambda_1 \succ \lambda_2$ **iff** λ_1 **is above or on the right of** λ_2.

Definition

For integers $k_1 \geq 3, k_2, \dots, k_r \geq 2$, we define the **multiple Eisenstein series** by

$$G_{k_1,\dots,k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \cdots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}}\,.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the **harmonic product**, i.e. it is for example

$$G_4(\tau) \cdot G_3(\tau) = G_{4,3}(\tau) + G_{3,4}(\tau) + G_7(\tau)\,.$$

In depth one we have for $k \geq 3$

$$G_k(\tau) = \sum_{\substack{\lambda \in \mathbb{Z}\tau + \mathbb{Z} \\ \lambda \succ 0}} \frac{1}{\lambda^k} = \sum_{\substack{m>0 \\ \vee\, (m=0 \wedge n>0)}} \frac{1}{(m\tau + n)^k} = \zeta(k) + \sum_{m>0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}$$

In depth one we have for $k \geq 3$

$$G_k(\tau) = \sum_{\substack{\lambda \in \mathbb{Z}\tau + \mathbb{Z} \\ \lambda \succ 0}} \frac{1}{\lambda^k} = \sum_{\substack{m>0 \\ \vee\, (m=0 \wedge n>0)}} \frac{1}{(m\tau+n)^k} = \zeta(k) + \sum_{m>0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau+n)^k}$$

By the Lipschitz summation formula we get for $k \geq 2 \quad (q = e^{2\pi i \tau})$

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d>0} d^{k-1} q^d \qquad .$$

This gives

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{m>0 \\ d>0}} d^{k-1} q^{md} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

What is a multiple version of the divisor sum?

$$g(k) = \frac{1}{(k-1)!} \sum_{\substack{m>0 \\ d>0}} d^{k-1} q^{md} = \frac{1}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \,.$$

Definition

For $k_1, \dots k_r \geq 1$ we define the q-series $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$ by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ d_1, \dots, d_r > 0}} \frac{d_1^{k_1 - 1}}{(k_1 - 1)!} \dots \frac{d_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 d_1 + \dots + m_r d_r} \,.$$

These q-series have a nice combinatorial interpretation

$$g(k_1, \dots, k_r) = \sum_{n > 0} \left(\qquad \qquad \qquad \qquad \qquad \qquad \right) q^n \,.$$

Definition

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$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ d_1, \dots, d_r > 0}} \frac{d_1^{k_1 - 1}}{(k_1 - 1)!} \cdots \frac{d_r^{k_r - 1}}{(k_r - 1)!} q^{m_1 d_1 + \dots + m_r d_r} .$$

Proposition (g are q-analogues of MZVs)

For $k_1 \geq 2, k_2 \dots, k_r \geq 1$ we have

$$\lim_{q \to 1} (1 - q)^{k_1 + \dots + k_r} g(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r) .$$

③ Multiple Eisenstein series - Fourier expansion

Theorem (Gangl-Kaneko-Zagier 2006 ($r = 2$), B. 2012 ($r \geq 2$))

The multiple Eisenstein series $G_{k_1,\ldots,k_r}(\tau)$ have a Fourier expansion of the form

$$G_{k_1,\ldots,k_r}(\tau) = \zeta(k_1,\ldots,k_r) + \sum_{n>0} a_n q^n \qquad \left(q = e^{2\pi i\tau}\right)$$

and they can be written explicitly as a $\mathcal{Z}[2\pi i]$-linear combination of q-analogues of multiple zeta values g. In particular, $a_n \in \mathcal{Z}[2\pi i]$.

Examples

$$G_k(\tau) = \zeta(k) + (-2\pi i)^k g(k)\,,$$

$$G_{3,2}(q) = \zeta(3,2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3,2)\,.$$

We saw the following example:

Example

$$\begin{aligned}\zeta(2)\cdot\zeta(3) &\overset{\text{harmonic}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)\\ &\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)\,.\end{aligned}$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

Question

Are these relations also satisfied by the multiple Eisenstein series?

We saw the following example:

Example

$$\zeta(2)\cdot\zeta(3) \overset{\text{harmonic}}{=} \zeta(2,3)+\zeta(3,2)+\zeta(5)$$
$$\overset{\text{shuffle}}{=} \zeta(2,3)+3\zeta(3,2)+6\zeta(4,1)\,.$$

$$\Longrightarrow 2\zeta(3,2)+6\zeta(4,1) \overset{\text{double shuffle}}{=} \zeta(5)\,.$$

Question

Are these relations also satisfied by the multiple Eisenstein series?

Problem: No definition of $G_{2,3}$ and $G_{4,1}$!

③ Multiple Eisenstein series - Do they satisfy double shuffle?

There are different ways to extend the definition of $G_{k_1,\dots,k_r}$ to $k_1,\dots,k_r \geq 1$

- Formal double zeta space realization $G_{r,s}$ (Gangl-Kaneko-Zagier, 2006)

$$G_{k_1} \cdot G_{k_2} + (\delta_{k_1,2} + \delta_{k_2,2}) \frac{G'_{k_1+k_2-2}}{2(k_1+k_2-2)} = G_{k_1,k_2} + G_{k_2,k_1} + G_{k_1+k_2}$$
$$= \sum_{j=2}^{k_1+k_2-1} \left(\binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) G_{j,k_1+k_2-j}\,, \qquad (k_1+k_2 \geq 3)\,.$$

- Finite double shuffle version $G_{r,s}$ (Kaneko, 2007).
- Shuffle regularized multiple Eisenstein series $G^{\sqcup\!\sqcup}_{k_1,\dots,k_r}$ (B.-Tasaka, 2017).
- Harmonic regularized multiple Eisenstein series $G^{*}_{k_1,\dots,k_r}$ (B., 2019).

Observation

- No version of these objects satisfy the double shuffle relations for all indices/weights.
- The derivative is always somewhere as an extra term.

③ Multiple Eisenstein series - Of course, they do not satisfy dsh...

Theorem (Gangl-Kaneko-Zagier + ϵ)

For all $k \geq 0$ there exists a basis of S_k given by explicit linear combinations of $G_{\text{odd},\text{odd}}$.

Corollary (taking constant term)

For each cusp form there is a relation among $\zeta(\text{odd}, \text{odd})$.

Example There exist a $c \in \mathbb{C}$ with

$$c\Delta = G_{3,9} - \frac{23825}{5197} G_{5,7} - \frac{41431}{10394} G_{7,5} + \frac{360}{5197} G_{9,3} + G_{11,1}\,,$$

which implies the relation

$$0 = \zeta(3,9) - \frac{23825}{5197}\zeta(5,7) - \frac{41431}{10394}\zeta(7,5) + \frac{360}{5197}\zeta(9,3) + \zeta(11,1)\,.$$

Conjecturally these are the only relations among $\zeta(\text{odd}, \text{odd})$

$\rightsquigarrow$ Explanation of $\mathsf{O}(X)\,\mathsf{O}(X) - \mathsf{S}(X)$ in the Broadhurst-Kreimer conjecture.

Questions

- What are the relations satisfied by multiple Eisenstein series?
- Can we formalize these relation?

The double shuffle relations can also be stated in terms of **generating series:**

$$Z^*(X_1,\dots,X_r) = \sum_{k_1,\dots,k_r\geq 1} \zeta^*(k_1,\dots,k_r) X_1^{k_1-1}\dots X_r^{k_r-1}$$

Then the extended double shuffle relations in lowest depths can be written as

$$\begin{aligned} Z^*(X_1)Z^*(X_2) &= Z^*(X_1,X_2) + Z^*(X_2,X_1) + \frac{Z^*(X_1)-Z^*(X_2)}{X_1-X_2} \\ &= Z^*(X_1+X_2,X_2) + Z^*(X_1+X_2,X_1) + \zeta(2)\,. \end{aligned}$$

- A: $\mathbb{Q}$-algebra.
- For $z_{k_1,\dots,k_r} \in A$ for $k_1,\dots,k_r \geq 1$ we write

$$Z(X_1,\dots,X_r) = \sum_{k_1,\dots,k_r \geq 1} z_{k_1,\dots,k_r} X_1^{k_1-1} \dots X_r^{k_r-1}\,.$$

- A collection $Z = (Z(X_1,\dots,X_r))_{r\geq 1}$ will be called a **mould**.

Definition

A mould Z satisfies the **double shuffle relations** (in depth 2) if

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1,X_2) + Z(X_2,X_1) + \frac{Z(X_1)-Z(X_2)}{X_1-X_2} \\ &= Z(X_1+X_2,X_1) + Z(X_1+X_2,X_2) + z_2\,. \end{aligned}$$

- $A = \mathbb{R}$: Harmonic regularized multiple zeta values

$$z_{k_1,\ldots,k_r} = \zeta^*(k_1,\ldots,k_r)\,.$$

- $A = \mathbb{Q}$: Explicit solutions are known up to depth 3 (Brown, Ecalle, Gangl-Kaneko-Zagier, Tasaka). In depth 1 they are all given by

$$z_k = \begin{cases} -\frac{B_k}{2k!} = \frac{\zeta(k)}{(2\pi i)^k}\,, & k \text{ even} \\ 0\,, & k \text{ odd} \end{cases}\,.$$

- $A = \mathbb{Q}$: Solution exist in all depths (Drinfel'd + Furusho, Racinet).

General idea

- Include also (arbitrary) derivatives as objects.
- Instead of series $Z(X_1, \ldots, X_r)$ we will consider generating series with two types of variables X_i and Y_i.
- Roughly: X_i: weight, Y_i: derivative.
- In the case $Y_i = 0$, we get back our original story.

A: $\mathbb{Q}$-algebra

$$B\begin{pmatrix} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{pmatrix} \in A[[X_1, Y_1, \ldots, X_r, Y_r]].$$

Definition

A collection $B = \left(B\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r} \right)_{r \geq 1}$ will be called a **bimould**.

Definition

A bimould B is **symmetril** (up to depth 2), if

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2}=B\binom{X_1,X_2}{Y_1,Y_2}+B\binom{X_2,X_1}{Y_2,Y_1}+\frac{B\binom{X_1}{Y_1+Y_2}-B\binom{X_2}{Y_1+Y_2}}{X_1-X_2}.$$

Remark

- Can be written down explicitly in arbitrary depth.
- This corresponds to the harmonic product of MZV, i.e. compare it to

$$Z(X_1)Z(X_2)=Z(X_1,X_2)+Z(X_2,X_1)+\frac{Z(X_1)-Z(X_2)}{X_1-X_2}.$$

Definition

A bimould B is called **swap invariant** if

$$B\begin{pmatrix} X_1,\dots,X_r \\ Y_1,\dots,Y_r \end{pmatrix} = B\begin{pmatrix} Y_1+\dots+Y_r, Y_1+\dots+Y_{r-1},\dots,Y_1+Y_2,Y_1 \\ X_r, X_{r-1}-X_r,\dots,X_2-X_3,X_1-X_2 \end{pmatrix}.$$

$$B\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \overset{\text{sw}}{=} B\begin{pmatrix} Y_1 \\ X_1 \end{pmatrix}, \qquad B\begin{pmatrix} X_1,X_2 \\ Y_1,Y_2 \end{pmatrix} \overset{\text{sw}}{=} B\begin{pmatrix} Y_1+Y_2,Y_1 \\ X_2,X_1-X_2 \end{pmatrix}.$$

Recall symmetrility and swap in low depth

$$B\binom{X_1}{Y_1} \overset{\text{SW}}{=} B\binom{Y_1}{X_1}, \qquad B\binom{X_1, X_2}{Y_1, Y_2} \overset{\text{SW}}{=} B\binom{Y_1+Y_2, Y_1}{X_2, X_1-X_2},$$

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} \overset{\text{IL}}{=} B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1-X_2}.$$

Definition

Swap + Symmetril + Swap = **q-shuffle**

$$\begin{aligned}
B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} &\overset{\text{SW}}{=} B\binom{Y_1}{X_1}B\binom{Y_2}{X_2}\\
&\overset{\text{IL}}{=} B\binom{Y_1, Y_2}{X_1, X_2} + B\binom{Y_2, Y_1}{X_2, X_1} + \frac{B\binom{Y_1}{X_1+X_2} - B\binom{Y_2}{X_1+X_2}}{Y_1-Y_2}\\
&\overset{\text{SW}}{=} B\binom{X_1+X_2, X_1}{Y_2, Y_1-Y_2} + B\binom{X_1+X_2, X_2}{Y_1, Y_2-Y_1} + \frac{B\binom{X_1+X_2}{Y_1} - B\binom{X_1+X_2}{Y_2}}{Y_1-Y_2}.
\end{aligned}$$

Definition

A bimould satisfies **q-double shuffle** (in depth 2) if

$$B\binom{X_1}{Y_1}B\binom{X_2}{Y_2} = B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}$$
$$= B\binom{X_1+X_2, X_1}{Y_2, Y_1 - Y_2} + B\binom{X_1+X_2, X_2}{Y_1, Y_2 - Y_1} + \frac{B\binom{X_1+X_2}{Y_1} - B\binom{X_1+X_2}{Y_2}}{Y_1 - Y_2},$$

i.e. B is symmetril and satisfies the q-shuffle product formula.

- Clearly: Symmetril + Swap invariant $\Longrightarrow$ q-double shuffle.
- Compare this to the double shuffle relations

$$Z(X_1)Z(X_2) = Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}$$
$$= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2\,.$$

Solution to q-dsh => solution to dsh

Proposition

Let B be symmetril and swap invariant with $\frac{d}{dX}\frac{d}{dY}B\binom{X}{Y} = 0$. Then

$$Z(X) = B\begin{pmatrix} X \\ 0 \end{pmatrix}, \quad Z(X_1, X_2) = B\begin{pmatrix} X_1, X_2 \\ 0, 0 \end{pmatrix}$$

satisfies the double shuffle relations.

Proof:

$$\frac{B\binom{X_1+X_2}{Y_1} - B\binom{X_1+X_2}{Y_2}}{Y_1 - Y_2}_{|Y_1=Y_2=0} = \sum_{k\geq 1} b\begin{pmatrix} k \\ 1 \end{pmatrix}(X_1+X_2)^{k-1}$$

$$= b\begin{pmatrix} 1 \\ 1 \end{pmatrix} \overset{\text{sw}}{=} b\begin{pmatrix} 2 \\ 0 \end{pmatrix} = z_2 \,.$$

Interpretation: "When the derivative vanishes (constant function) then we obtain a solution to classical dsh (equations for numbers)".

Theorem (B.-Kühn-Matthes, 2020+)

Let Z satisfy the double shuffle relations (in all depths), then there exists an explicit construction of a symmetril and swap invariant bimould B.

For example, in lowest depth the bimould

$$B\binom{X_1}{Y_1} = Z(X_1) + Z(Y_1)\,,$$

$$B\binom{X_1, X_2}{Y_1, Y_2} = Z(X_1, X_2) + Z(Y_1 + Y_2, Y_1) + Z(X_2)Z(Y_1) + \frac{1}{2}z_2$$

is symmetril and swap invariant

Interpretation: "Numbers can be viewed as constant functions".

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

There exist a symmetril and swap invariant bimould $\mathfrak{G}$ (up to depth 3) which in depth one is given by the generating series of derivatives of Eisenstein series.

Remark

- This setup gives combinatorial proofs of classical identities (e.g. Ramanujan Diff.eq.).
- The construction of this bimould is inspired by the Fourier expansion of multiple Eisenstein series.
- The bimould $\mathfrak{G}$ can be written down explicitly in terms of rational solutions to the classical double shuffle equations and a bi-variant of the q-series g.
- We have a conjectured construction for all depths (j.w. A. Burmester).

Definition

We define the **combinatorial multiple Eisenstein series** G in depth ≤ 2 by

$$\mathfrak{G}\binom{X}{Y} =: \sum_{\substack{k\geq 1\\ d\geq 0}} G\binom{k}{d} X^{k-1}\frac{Y^d}{d!}\,,$$

$$\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} =: \sum_{\substack{k_1,k_2\geq 1\\ d_1,d_2\geq 0}} G\binom{k_1, k_2}{d_1, d_2} X_1^{k_1-1}X_2^{k_2-1}\frac{Y_1^{d_1}}{d_1!}\frac{Y_2^{d_2}}{d_2!}\,.$$

- In depth one $G\binom{k}{d}$ is basically the d-th derivative of G_{k-d}.
- In depth two the $G\binom{k_1,k_2}{0,0}$ are (almost) the double Eisenstein series.
- The symmetrility $\mathfrak{G}$ of gives

$$G\binom{k_1}{d_1}G\binom{k_2}{d_2} = G\binom{k_1, k_2}{d_1, d_2} + G\binom{k_2, k_1}{d_2, d_1} + G\binom{k_1+k_2}{d_1+d_2}\,.$$

Definition

Space of double combinatorial multiple Eisenstein series of weight $K \geq 1$:

$$\mathfrak{D}_K = \left\langle G\binom{k}{d}, G\binom{k_1, k_2}{d_1, d_2} \;\middle|\; \substack{k+d=k_1+k_2+d_1+d_2=K \\ k,k_1,k_2\geq 1,\, d,d_1,d_2\geq 0} \right\rangle_{\mathbb{Q}}$$

Proposition

$$q\frac{d}{dq}\mathfrak{G}\binom{X_1}{Y_1} = \frac{d}{dX_1}\frac{d}{dY_1}\mathfrak{G}\binom{X_1}{Y_1},$$

$$q\frac{d}{dq}\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} = \left(\frac{d}{dX_1}\frac{d}{dY_1} + \frac{d}{dX_2}\frac{d}{dY_2}\right)\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2}.$$

Corollary

Combinatorial multiple Eisenstein series are closed under $q\frac{d}{dq}$. In particular

$$q\frac{d}{dq}\mathfrak{D}_K \subset \mathfrak{D}_{K+2}.$$

$$\mathfrak{D}_K = \left\langle G\binom{k}{d}, G\binom{k_1,k_2}{d_1,d_2} \middle| \begin{smallmatrix} k+d=k_1+k_2+d_1+d_2=K \\ k,k_1,k_2\geq 1,\, d,d_1,d_2\geq 0 \end{smallmatrix} \right\rangle_{\mathbb{Q}}$$

$$\mathfrak{D}_K^0 = \left\langle G\binom{k}{0}, G\binom{k_1,k_2}{0,0} \in \mathfrak{D}_K \right\rangle_{\mathbb{Q}}$$

Proposition

- $\mathfrak{D}_K$ contains the space of **quasi modular forms** $\mathbb{Q}[\tilde{G}_2, \tilde{G}_4, \tilde{G}_6]_K$ of weight K.
- $\mathfrak{D}_K^0$ contains the space of **modular forms** $\mathbb{Q}[\tilde{G}_4, \tilde{G}_6]_K$ of weight K

Numerical computer calculation give:

k	1	2	3	4	5	6	7	8
$\dim \mathfrak{D}_K \overset{?}{=}$	1	2	3	5	7	11	14	..
$\dim \mathfrak{D}_K^0 \overset{?}{=}$	1	2	3	3	4	4	5	5
# generators of $\mathfrak{D}_K$	1	3	7	14	25	41	63	92

	Numbers	Functions / q-series
"single" objects	Riemann zeta values $\zeta(k)$	Eisenstein series $G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n\geq 1} \frac{n^{k-1}q^n}{1-q^n}$ $(q = e^{2\pi i\tau})$
"multiple" objects	Multiple zeta values $\zeta(k_1, \dots, k_r)$	Multiple Eisenstein series $G_{k_1,\dots,k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n>0} a_n q^n$
relations	Double shuffle relations	Symmetril & Swap invariant
object point of view	$\zeta(2) \cdot \zeta(3) = \zeta(2,3) + \zeta(3,2) + \zeta(5)$ $= \zeta(2,3) + 3\zeta(3,2) + 6\zeta(4,1)$	
generating series point of view	$Z(X_1)Z(X_2) = Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}$ $= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2\,.$	$B\binom{X_1}{Y_1} \overset{\mathrm{sw}}{=} B\binom{Y_1}{X_1}, \quad B\binom{X_1, X_2}{Y_1, Y_2} \overset{\mathrm{sw}}{=} B\binom{Y_1 + Y_2, Y_1}{X_2, X_1 - X_2},$ $B\binom{X_1}{Y_1} B\binom{X_2}{Y_2} \overset{\mathrm{IL}}{=} B\binom{X_1, X_2}{Y_1, Y_2} + B\binom{X_2, X_1}{Y_2, Y_1} + \frac{B\binom{X_1}{Y_1+Y_2} - B\binom{X_2}{Y_1+Y_2}}{X_1 - X_2}.$

⑤ Bonus - Basis conjecture

Conjecture (Hoffman, 1997)

For $k \geq 0$ the multiple zeta values

$$\{\zeta(k_1, \ldots, k_r) \mid r \geq 0,\ k_1 + \cdots + k_r = k,\ k_1, \ldots, k_r \in \{2,3\}\}$$

form a basis of $\mathcal{Z}_k$.

$$\zeta(2n) \in \pi^{2n}\mathbb{Q}, \qquad \zeta(2, \ldots, 2) = \frac{\pi^{2n}}{(2n+1)!}, \quad \zeta(5) = \frac{6}{5}\zeta(2,3) + \frac{4}{5}\zeta(3,2)\,.$$

Theorem (Brown, 2012)

For all $k \geq 0$ we have

$$\mathcal{Z}_k = \langle \zeta(k_1, \ldots, k_r) \mid r \geq 0,\ k_1 + \cdots + k_r = k,\ k_1, \ldots, k_r \in \{2,3\} \rangle_{\mathbb{Q}}\,.$$

Definition

A bimould B is **symmetril** (up to depth 3), if

$$B\begin{pmatrix}X_1\\Y_1\end{pmatrix}B\begin{pmatrix}X_2\\Y_2\end{pmatrix}=B\begin{pmatrix}X_1,X_2\\Y_1,Y_2\end{pmatrix}+B\begin{pmatrix}X_2,X_1\\Y_2,Y_1\end{pmatrix}+\frac{B\left(\begin{smallmatrix}X_1\\Y_1+Y_2\end{smallmatrix}\right)-B\left(\begin{smallmatrix}X_2\\Y_1+Y_2\end{smallmatrix}\right)}{X_1-X_2},$$

$$\begin{aligned}B\begin{pmatrix}X_1\\Y_1\end{pmatrix}B\begin{pmatrix}X_2,X_3\\Y_2,Y_3\end{pmatrix}&=B\begin{pmatrix}X_1,X_2,X_3\\Y_1,Y_2,Y_3\end{pmatrix}+B\begin{pmatrix}X_2,X_1,X_3\\Y_2,Y_1,Y_3\end{pmatrix}+B\begin{pmatrix}X_2,X_3,X_1\\Y_2,Y_3,Y_1\end{pmatrix}\\&+\frac{B\left(\begin{smallmatrix}X_1,X_3\\Y_1+Y_2,Y_3\end{smallmatrix}\right)-B\left(\begin{smallmatrix}X_2,X_3\\Y_1+Y_2,Y_3\end{smallmatrix}\right)}{X_1-X_2}+\frac{B\left(\begin{smallmatrix}X_2,X_1\\Y_2,Y_1+Y_3\end{smallmatrix}\right)-B\left(\begin{smallmatrix}X_2,X_3\\Y_2,Y_1+Y_3\end{smallmatrix}\right)}{X_1-X_3}.\end{aligned}$$

⑤ Bonus - The bimould $\mathfrak{g}$

Define for $m \geq 1$ the series

$$L_m\begin{pmatrix} X \\ Y \end{pmatrix} = \frac{e^{X+mY}q^m}{1-e^X q^m} = \sum_{n\geq 1} e^{nX+mY}q^{mn}\,.$$

Definition

We define the bimould $\mathfrak{g}$ for all depth $r \geq 1$ by

$$\mathfrak{g}\begin{pmatrix} X_1,\dots,X_r \\ Y_1,\dots,Y_r \end{pmatrix} = \sum_{m_1>\cdots>m_r>0} L_{m_1}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}\dots L_{m_r}\begin{pmatrix} X_r \\ Y_r \end{pmatrix}.$$

Proposition

The bimould $\mathfrak{g}$ is swap invariant.

Example: Swap invariance of $\mathfrak{g}$ in depth 2

$$\mathfrak{g}\begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} = \sum_{m_1>m_2>0} L_{m_1}\begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} L_{m_2}\begin{pmatrix} X_2 \\ Y_2 \end{pmatrix}$$

$$= \sum_{\substack{m_1>m_2>0 \\ n_1,n_2>0}} e^{n_1X_1+n_2X_2+m_1Y_1+m_2Y_2} q^{m_1n_1+m_2n_2} = (\star)$$

Change of variables $\leftrightsquigarrow$ swap of variables

$$\left\{ \begin{array}{ll} m_1 = m_1' + m_2' & , m_2 = m_1' \\ n_1 = n_2' & , n_2 = n_1' - n_2' \end{array} \right\} \Longrightarrow m_1n_1 + m_2n_2 = m_1'n_1' + m_2'n_2'.$$

$$(\star) = \sum_{\substack{m_1',m_2'>0 \\ n_1'>n_2'>0}} e^{n_2'X_1+(n_1'-n_2')X_2+(m_1'+m_2')Y_1+m_1'Y_2} q^{m_1'n_1'+m_2'n_2'}$$

$$= \mathfrak{g}\begin{pmatrix} Y_1+Y_2, Y_1 \\ X_2, X_1-X_2 \end{pmatrix}.$$

⑤ Bonus - Product of $\mathfrak{g}$

$$
\begin{aligned}
\mathfrak{g}\binom{X_1}{Y_1}\mathfrak{g}\binom{X_2}{Y_2} &= \sum_{m_1>0} L_{m_1}\binom{X_1}{Y_1} \sum_{m_2>0} L_{m_2}\binom{X_2}{Y_2} \\
&= \left(\sum_{m_1>m_2>0} + \sum_{m_2>m_1>0} + \sum_{m_1=m_2>0}\right) L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2} \\
&= \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1} + \sum_{m>0} L_m\binom{X_1}{Y_1} L_m\binom{X_2}{Y_2}.
\end{aligned}
$$

To describe the product of $\mathfrak{g}$ we need to describe for a fixed m the product of L_m.

$$\mathfrak{g}\binom{X_1}{Y_1}\mathfrak{g}\binom{X_2}{Y_2} = \sum_{m_1>0} L_{m_1}\binom{X_1}{Y_1}\sum_{m_2>0} L_{m_2}\binom{X_2}{Y_2}$$
$$= \left(\sum_{m_1>m_2>0} + \sum_{m_2>m_1>0} + \sum_{m_1=m_2>0}\right) L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2}$$
$$= \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}\binom{X_2, X_1}{Y_2, Y_1} + \sum_{m>0} L_m\binom{X_1}{Y_1} L_m\binom{X_2}{Y_2}.$$

Lemma

For all $m \geq 1$ we have

$$L_m\binom{X_1}{Y_1}L_m\binom{X_2}{Y_2} = \frac{L_m\binom{X_1}{Y_1+Y_2} - L_m\binom{X_2}{Y_1+Y_2}}{X_1 - X_2} + L_m\binom{X_1, X_2}{Y_1, Y_2} + L_m\binom{X_2, X_1}{Y_2, Y_1}$$

where

$$L_m\binom{X_1, X_2}{Y_1, Y_2} = L_m\binom{X_1}{Y_1+Y_2}\left(\beta\binom{X_2 - X_1}{-Y_2} - \frac{1}{2}\right) + \beta\binom{X_1 - X_2}{Y_1} L_m\binom{X_2}{Y_1+Y_2}$$

⑤ Bonus - Combinatorial MES

Proposition

For all $m \geq 1$ the series

$$L_m\binom{X}{Y} = \frac{e^{X+mY} q^m}{1 - e^X q^m} ,$$

$$L_m\binom{X_1, X_2}{Y_1, Y_2} = L_m\binom{X_1}{Y_1 + Y_2}\left(\beta\binom{X_2 - X_1}{-Y_2} - \frac{1}{2}\right) + \beta\binom{X_1 - X_2}{Y_1} L_m\binom{X_2}{Y_1 + Y_2},$$

$$L_m\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} = \text{explicit long formula}$$

are symmetril.

Remark

- The $L_m\binom{X}{Y}$ can be seen as the the generating series of "bi-monotangent" function.
- The construction of $L_m\binom{X_1,\ldots,X_r}{Y_1,\ldots,Y_r}$ in terms of β and $L_m\binom{X}{Y}$ corresponds to "Multitangent = MZV-linear combination of monotangent".

⑤ Bonus - Make $\mathfrak{g}$ symmetril

Proposition

If L_m is symmetril for all $m \geq 1$, then

$$\mathfrak{g}^{\mathrm{il}}\binom{X_1}{Y_1} = \sum_{m>0} L_m\binom{X_1}{Y_1},$$

$$\mathfrak{g}^{\mathrm{il}}\binom{X_1, X_2}{Y_1, Y_2} = \sum_{m_1>m_2>0} L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2} + \sum_{m>0} L_m\binom{X_1, X_2}{Y_1, Y_2},$$

$$\begin{aligned}\mathfrak{g}^{\mathrm{il}}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} &= \sum_{m_1>m_2>m_3>0} L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2}{Y_2} L_{m_3}\binom{X_3}{Y_3} \\ &+ \sum_{m_1>m_2>0} \left(L_{m_1}\binom{X_1, X_2}{Y_1, Y_2} L_{m_2}\binom{X_3}{Y_3} + L_{m_1}\binom{X_1}{Y_1} L_{m_2}\binom{X_2, X_3}{Y_2, Y_3} \right) \\ &+ \sum_{m>0} L_m\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3},\end{aligned}$$

are also symmetril.

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

The following series are symmetril and swap invariant

$$\mathfrak{G}\binom{X_1}{Y_1} = \mathfrak{g}^{\mathrm{il}}\binom{X_1}{Y_1} + \beta\binom{X_1}{Y_1},$$

$$\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} = \mathfrak{g}^{\mathrm{il}}\binom{X_1, X_2}{Y_1, Y_2} + \mathfrak{g}^{\mathrm{il}}\binom{X_2}{Y_2}\beta\binom{X_1}{Y_1} + \beta\binom{X_1, X_2}{Y_1, Y_2},$$

$$\begin{aligned}\mathfrak{G}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} = {} & \mathfrak{g}^{\mathrm{il}}\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3} + \mathfrak{g}^{\mathrm{il}}\binom{X_1, X_2}{Y_1, Y_2}\beta\binom{X_3}{Y_3} \\ & + \mathfrak{g}^{\mathrm{il}}\binom{X_1}{Y_1}\beta\binom{X_2, X_3}{Y_2, Y_3} + \beta\binom{X_1, X_2, X_3}{Y_1, Y_2, Y_3}.\end{aligned}$$

In the mould language: $\mathfrak{G}$ is the mould product of the two symmetril bimoulds $\mathfrak{g}^{\mathrm{il}}$ and β.

⑤ Bonus - Combinatorial MES explicit

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

The following series are symmetril and swap invariant

$$\mathfrak{G}\binom{X_1}{Y_1} = \beta\binom{X_1}{Y_1} + \mathfrak{g}\binom{X_1}{Y_1},$$

$$\begin{aligned}\mathfrak{G}\binom{X_1, X_2}{Y_1, Y_2} &= \beta\binom{X_1, X_2}{Y_1, Y_2} - \beta\binom{X_1 - X_2}{Y_2}\mathfrak{g}\binom{X_1}{Y_1 + Y_2} - \frac{1}{2}\mathfrak{g}\binom{X_1}{Y_1 + Y_2} \\ &\quad + \beta\binom{X_2}{Y_2}\mathfrak{g}\binom{X_1}{Y_1} + \beta\binom{X_1 - X_2}{Y_1}\mathfrak{g}\binom{X_2}{Y_1 + Y_2} + \mathfrak{g}\binom{X_1, X_2}{Y_1, Y_2}.\end{aligned}$$

- In the mould language: $\mathfrak{G}$ is the mould product of the two symmetril bimoulds $\mathfrak{g}^{\mathrm{il}}$ and β.
- For the construction for depth ≥ 3 (which conjecturally works in all depths) see the talkslides of Annika Burmesters talk "Combinatorial multiple Eisenstein series" at the JENTE Seminar (https://sites.google.com/view/jente-seminar/home).