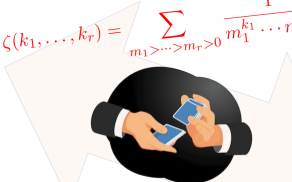


# A variant of the double shuffle relations and quasi modular forms

**Henrik Bachmann**

Nagoya University



The illustration shows a person in a black suit and white shirt, seen from the chest up, shuffling a deck of blue cards. The person is positioned in the center of a large, light orange, four-pointed star shape. Overlaid on the top-left arm of the star is the red formula  $\zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}$ . Overlaid on the bottom-right arm of the star is the green formula  $\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$ .

$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}}$$
$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24}$$

j.w. Annika Burmester

Ulf Kühn & Nils Matthes

ACPMs Seminar, 4th December 2020

[www.henrikbachmann.com](http://www.henrikbachmann.com)

# Lecture with notes & videos related to this talk

<https://www.henrikbachmann.com/mzv2020.html>

Henrik Bachmann

- Multiple Eisenstein series [B2], [GKZ]
- $q$ -analogues of multiple zeta values [B2]

Materials

- Lecture notes & Exercises: Multiple zeta values & modular forms [ver. 5.2]
- Lecture videos: Lecture 1, Lecture 2, Lecture 3, Lecture 4, Lecture 5, Lecture 6, Lecture 7, Lecture 8, Lecture 9, Lecture 10, Lecture 11, Lecture 12, Lecture 13, Lecture 14, Lecture 15
- Zoom meeting notes: Meeting 2 (05/01), Meeting 3 (05/08), Meeting 4 (05/15), Meeting 5 (05/22), Meeting 6 (05/29), Meeting 7 (06/05), Meeting 8 (06/12), Meeting 9 (06/19), Meeting 10 (06/26), Meeting 11 (07/03), Meeting 12 (07/10), Meeting 13 (07/17), Meeting 14 (07/24), Meeting 15 (07/31), Meeting 16 (08/07)
- Quasi-shuffle products playground [online tool]

Suchen

What objects will we study in this course?

Numbers  $\mathbb{R}$

$\zeta(k)$   
 $\zeta(k_1, \dots, k_r)$   
MZV

$q=0$

Functions  $q(k)$

Modular forms  $g(k)$   
 $g(k) = \zeta(k) + \sum_{n=1}^{\infty} a_n q^n$   
 $q = e^{2\pi i \tau}$

$g(k)$   
 $q$ -analogues of MZV

q-revise (GKZ)

Multiple zeta values & modular forms: Lecture 1 - Introduction & Riemann zeta function

484 Aufrufe · 29.04.2020

mzv\_mf\_2020.pdf  
Page 3 of 119

MZVs and modular forms • Overview & Basics

## §1 Overview & Basics

In this section, we will give an overview of the values of the Riemann zeta function, the definition of multiple zeta values, some of the main conjectures concerning their structure, and a glimpse of the connection of ( $q$ -analogues of) multiple zeta values and modular forms. The general picture of these concepts will be discussed in detail in the later Sections.

### 1.1 The values of the Riemann zeta function

The Riemann zeta function is defined for a complex variable  $s \in \mathbb{C}$  with  $\operatorname{Re}(s) > 1$  by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \quad (1.1)$$

This function appears in various fields of mathematics and theoretical physics and it can be studied from various points of view. It plays a pivotal role in analytic number theory and has applications in physics, probability theory, and applied statistics.

For example, it is well-known that the Riemann zeta function can be analytically continued to the whole complex plane with a simple pole at  $s = 1$ . Even though  $\zeta(s)$  was already considered by L. Euler (1707 – 1783), it was named after B. Riemann (1826 – 1866), who proved its meromorphic continuation and functional equation and established a relation between its zeros and the distribution of prime numbers. In particular, he gave his famous conjecture on the location of the zeros of the Riemann zeta function, stating that besides the trivial zeros at  $s = -2, -4, -6, \dots$  all other zeros have real part  $\frac{1}{2}$ .

The graph of  $|\zeta(s + yi)|$  near the pole at  $s = 1 + yi = 1$ .

The connection to prime numbers is given by the following product formula, which, to make things fair again, was named after Euler (Euler product formula)

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}} \quad (\operatorname{Re}(s) > 1)$$

sets then the

A sculpture of the pole of  $\zeta(s)$  at  $s = 1$  in front of Nagoya station in honor of the

Why "shuffle" & "shuffle" ?

(harmonic)

$\zeta(k_1)\zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2)$

"MZV" = Deck of cards

$\zeta(k_1)\zeta(k_2) = \sum_{j=0}^{k_1+k_2-1} \left( \binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j)$

$\zeta(k_1) = \sum_{j=1}^k \frac{1}{j^{k_1}}$

$\zeta(k_1) + \zeta(k_2) = \sum_{j=1}^{k_1+k_2} \frac{1}{j^{k_1+k_2}}$

$\zeta(k_1) + \zeta(k_2) = \zeta(k_1+k_2)$

Numbers

Functions / q-series

“single”  
objects

“multiple”  
objects

relations

## Numbers

## Functions / q-series

“single” objects	Riemann zeta values $\zeta(k)$	
“multiple” objects	Multiple zeta values $\zeta(k_1, \dots, k_r)$	
relations		



## Numbers

## Functions / q-series

“single” objects	Riemann zeta values $\zeta(k)$	
“multiple” objects	Multiple zeta values $\zeta(k_1, \dots, k_r)$	
relations	Double shuffle relations $\begin{aligned}\zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1)\end{aligned}$	

# Numbers

# Functions / q-series

<b>“single” objects</b>	<b>Riemann zeta values</b> $\zeta(k)$	
<b>“multiple” objects</b>	<b>Multiple zeta values</b> $\zeta(k_1, \dots, k_r)$	
<b>relations</b>  <b>object point of view</b>  <b>generating series point of view</b>	<b>Double shuffle relations</b>  $\begin{aligned}\zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1)\end{aligned}$ $\begin{aligned}Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2.\end{aligned}$	

# Numbers

# Functions / q-series

“single”  
objects

Riemann zeta values

$$\zeta(k)$$

Eisenstein series

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n}$$

$$(q = e^{2\pi i \tau})$$

“multiple”  
objects

Multiple zeta values

$$\zeta(k_1, \dots, k_r)$$

relations

Double shuffle relations

object point  
of view

$$\begin{aligned} \zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1) \end{aligned}$$

generating  
series point  
of view

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

# Numbers

# Functions / q-series

<p>“single” objects</p>	<p>Riemann zeta values</p> $\zeta(k)$	<p>Eisenstein series</p> $G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n}$ $(q = e^{2\pi i \tau})$
<p>“multiple” objects</p>	<p>Multiple zeta values</p> $\zeta(k_1, \dots, k_r)$	<p>Multiple Eisenstein series</p> $G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n > 0} a_n q^n$
<p>relations</p> <p>object point of view</p> <p>generating series point of view</p>	<p>Double shuffle relations</p> $\begin{aligned} \zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1) \end{aligned}$ $\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$	

# Numbers

# Functions / q-series

“single”  
objects

Riemann zeta values

$$\zeta(k)$$

Eisenstein series

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n}$$

$$(q = e^{2\pi i \tau})$$

“multiple”  
objects

Multiple zeta values

$$\zeta(k_1, \dots, k_r)$$

Multiple Eisenstein series

$$G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n > 0} a_n q^n$$

relations

Double shuffle relations

object point  
of view

$$\begin{aligned} \zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1) \end{aligned}$$

generating  
series point  
of view

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$



# Plan of this talk

1

Multiple zeta values  
&  
Double shuffle relations

3

Multiple Eisenstein series

1

Numbers

Functions / q-series

"single" objects	Riemann zeta values $\zeta(k)$	Eisenstein series $G_k(\tau) = \zeta(k) + \frac{(1-i\sqrt{3})^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1-q^n}$ ( $q = e^{2\pi i \tau}$ )
"multiple" objects	Multiple zeta values $\zeta(k_1, \dots, k_r)$	Multiple Eisenstein series $\mathcal{G}_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n \geq 0} a_n q^n$
relations	Double shuffle relations $\begin{aligned} \zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(5) \end{aligned}$	?
object point of view		
generating series point of view	$\begin{aligned} \zeta(X, 1)(X, y) &= 2\zeta(X, X, y) + 2\zeta(X, X, 1) + \frac{\partial \zeta(X, y)}{\partial X} \\ &= 2\zeta(X, X, X, y) + 2\zeta(X, X, X, 1) + \zeta_y \end{aligned}$	

2

3

4

2

Modular forms

4

"Double shuffle relations  
for functions"

## ① MZV & DSH - Definition

### Definition

For  $k_1 \geq 2, k_2, \dots, k_r \geq 1$  define the **multiple zeta value** (MZV)

$$\zeta(k_1, \dots, k_r) = \sum_{m_1 > \dots > m_r > 0} \frac{1}{m_1^{k_1} \dots m_r^{k_r}} \in \mathbb{R}.$$

By  $r$  we denote its **depth** and  $k_1 + \dots + k_r$  will be called its **weight**.

- $\mathcal{Z}$  :  $\mathbb{Q}$ -algebra of MZVs
- $\mathcal{Z}_k$ :  $\mathbb{Q}$ -vector space of MZVs of weight  $k$ .

MZVs can also be written as **iterated integrals**, e.g.

$$\zeta(2, 3) = \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{1-t_2} \int_0^{t_2} \frac{dt_3}{t_3} \int_0^{t_3} \frac{dt_4}{t_4} \int_0^{t_4} \frac{dt_5}{1-t_5}.$$

## ① MZV & DSH - Harmonic & shuffle product

There are two different ways to express the product of MZV in terms of MZV.

### Harmonic product (coming from the definition as iterated sums)

Example in depth two ( $k_1, k_2 \geq 2$ )

$$\begin{aligned}\zeta(k_1) \cdot \zeta(k_2) &= \sum_{m>0} \frac{1}{m^{k_1}} \sum_{n>0} \frac{1}{n^{k_2}} \\ &= \sum_{m>n>0} \frac{1}{m^{k_1} n^{k_2}} + \sum_{n>m>0} \frac{1}{m^{k_1} n^{k_2}} + \sum_{m=n>0} \frac{1}{m^{k_1+k_2}} \\ &= \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2) .\end{aligned}$$

### Shuffle product (coming from the expression as iterated integrals)

Example in depth two ( $k_1, k_2 \geq 2$ )

$$\zeta(k_1) \cdot \zeta(k_2) = \sum_{j=2}^{k_1+k_2-1} \left( \binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) \zeta(j, k_1 + k_2 - j) .$$



## ① MZV & DSH - Double shuffle relations

These two product expressions give various  $\mathbb{Q}$ -linear relations between MZV.

### Example

$$\begin{aligned}\zeta(2) \cdot \zeta(3) &\stackrel{\text{harmonic}}{=} \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &\stackrel{\text{shuffle}}{=} \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1) . \\ \implies 2\zeta(3, 2) + 6\zeta(4, 1) &\stackrel{\text{double shuffle}}{=} \zeta(5) .\end{aligned}$$

But there are more relations between MZV. e.g.:

$$\sum_{m>n>0} \frac{1}{m^2 n} = \zeta(2, 1) = \zeta(3) = \sum_{m>0} \frac{1}{m^3}.$$

These follow from regularizing the double shuffle relations

$\rightsquigarrow$  **extended double shuffle relations.**

## ① MZV & DSH - Relations conjectures

### Conjecture

All relations among MZVs are consequences of the extended double shuffle relations.

### Conjecture

The space  $\mathcal{Z}$  is graded by weight, i.e.

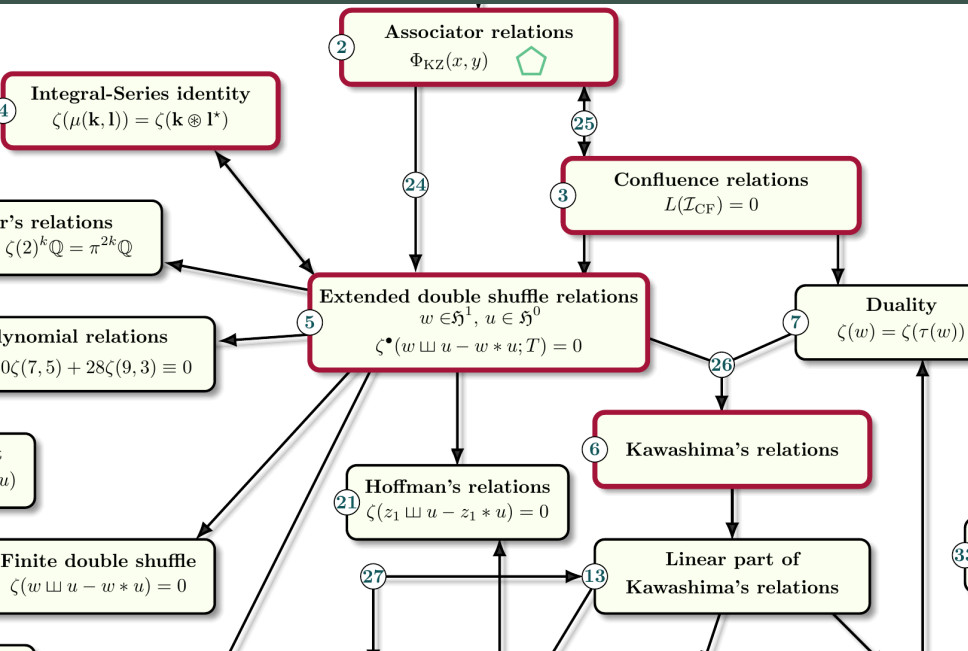
$$\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k .$$

- There are various different families of relations which conjecturally give all relations among MZV.
- Not for all of them it is known if they are equivalent to the extended double shuffle relations.

# Overview of relations among MZV

For details see:

B. "Multiple zeta values & modular forms", Lecture notes



## ① MZV & DSH - Dimension conjectures

Define the numbers  $d_k \in \mathbb{Z}_{\geq 0}$  by

$$\sum_{k \geq 0} d_k X^k = \frac{1}{1 - X^2 - X^3}.$$

### Conjecture (Zagier, 1994)

We have  $\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k$  for all  $k \geq 0$ .

weight $k$	0	1	2	3	4	5	6	7	8	9	10	11	12
# of adm. ind.	1	0	1	2	4	8	16	32	64	128	256	512	1024
# of relations $\stackrel{?}{=}$	0	0	0	1	3	6	14	29	60	123	249	503	1012
$d_k$	1	0	1	1	1	2	2	3	4	5	7	9	12

### Theorem (Terasoma (2002), Deligne–Goncharov (2005))

For all  $k \geq 0$  we have  $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$ .

## ② Modular forms - Definition

Complex upper half plane:  $\mathbb{H} = \{x + iy \in \mathbb{C} \mid x, y \in \mathbb{R}, y > 0\}$ .

### Definition

A holomorphic function  $f \in \mathcal{O}(\mathbb{H})$  is called a **modular form of weight**  $k \in \mathbb{Z}$  if it satisfies

- $f(\tau + 1) = f(\tau)$ ,
- $f(-\frac{1}{\tau}) = \tau^k f(\tau)$ ,

for all  $\tau \in \mathbb{H}$  and if it has a Fourier expansion of the form

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n. \quad (a_n \in \mathbb{C}, q = e^{2\pi i \tau})$$

- $\mathcal{M}_k$  : space of all modular forms of weight  $k$ .
- The space of **cusp forms** of weight  $k$  is defined by

$$\mathcal{S}_k = \{f \in \mathcal{M}_k \mid f = \sum_{\mathbf{n}=1}^{\infty} a_n q^n\}.$$

## ② Modular forms - Eisenstein series

For even  $k \geq 4$  the **Eisenstein series** are defined by

$$G_k(\tau) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(m\tau + n)^k}.$$

These have a Fourier expansion of the form

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n,$$

where  $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$  is the divisor sum.

### Proposition

For every even  $k \geq 4$  we have  $G_k \in \mathcal{M}_k$  and

$$\mathcal{M} = \bigoplus_{k=0}^{\infty} \mathcal{M}_k = \mathbb{C}[G_4, G_6].$$

## ② Modular forms - Quasi-modular forms

Are derivatives of modular forms again modular forms?... No

Define the **Eisenstein series of weight two** by

$$G_2(\tau) = \zeta(2) + (-2\pi i)^2 \sum_{n=1}^{\infty} \sigma_1(n) q^n ,$$

and the space of **quasi-modular forms** by

$$\widetilde{\mathcal{M}} = \mathbb{C}[G_2, G_4, G_6] .$$

### Proposition

The space  $\widetilde{\mathcal{M}}$  is the "smallest" ring with the following properties:

- It contains the ring of modular forms  $\mathcal{M}$ .
- It is closed under  $\frac{d}{d\tau}$ .

## ② Modular forms - Cusp forms

The first non-trivial cusp form is the **discriminant function**  $\Delta$

$$\Delta(\tau) = q \prod_{n=1}^{\infty} (1 - q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \dots,$$

which is a cusp form of weight 12.

### Theorem

- For  $k \geq 0$  the map  $\mathcal{M}_k \rightarrow \mathcal{S}_{k+12}$  given by  $f \mapsto \Delta \cdot f$  is an isomorphism of  $\mathbb{C}$ -vector spaces.
- The generating series for the dimension of cusp forms of weight  $k$  is given by

$$S(X) = \sum_{k \geq 0} \dim_{\mathbb{C}} \mathcal{S}_k X^k = X^{12} \sum_{k \geq 0} \dim_{\mathbb{C}} \mathcal{M}_k X^k = \frac{X^{12}}{(1 - X^4)(1 - X^6)}.$$



## ① ② MZV & Modular forms - Broadhurst-Kreimer conjecture

$\text{gr}_r^D \mathcal{Z}_k$ : MZV of weight  $k$  and depth  $r$  modulo lower depths MZV.

### Conjecture (Broadhurst-Kreimer, 1997)

The generating series of the dimensions of the weight- and depth-graded parts of multiple zeta values is given by

$$\sum_{k,r \geq 0} \dim_{\mathbb{Q}} (\text{gr}_r^D \mathcal{Z}_k) X^k Y^r = \frac{1 + E(X)Y}{1 - O(X)Y + S(X)Y^2 - S(X)Y^4},$$

where

$$E(X) = \frac{X^2}{1 - X^2}, \quad O(X) = \frac{X^3}{1 - X^2}, \quad S(X) = \frac{X^{12}}{(1 - X^4)(1 - X^6)}.$$

Observe that

$$\begin{aligned} & \frac{1 + E(X)Y}{1 - O(X)Y + S(X)Y^2 - S(X)Y^4} \\ &= 1 + (E(X) + O(X))Y + ((E(X) + O(X))O(X) - S(X))Y^2 + \dots \end{aligned}$$

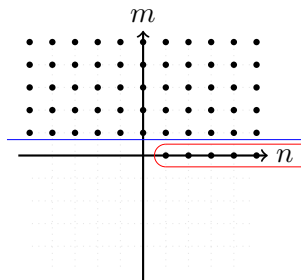
### ③ Multiple Eisenstein series - An order on lattices

Let  $\tau \in \mathbb{H}$ . We define an order  $\succ$  on the lattice  $\mathbb{Z}\tau + \mathbb{Z}$  by setting

$$\lambda_1 \succ \lambda_2 :\Leftrightarrow \lambda_1 - \lambda_2 \in P$$

for  $\lambda_1, \lambda_2 \in \mathbb{Z}\tau + \mathbb{Z}$  and the following set of positive lattice points

$$P := \{m\tau + n \in \mathbb{Z}\tau + \mathbb{Z} \mid m > 0 \vee (m = 0 \wedge n > 0)\} = U \cup R.$$



In other words:  $\lambda_1 \succ \lambda_2$  iff  $\lambda_1$  is **above** or on the **right** of  $\lambda_2$ .

### ③ Multiple Eisenstein series - Multiple Eisenstein series

#### Definition

For integers  $k_1 \geq 3, k_2, \dots, k_r \geq 2$ , we define the **multiple Eisenstein series** by

$$G_{k_1, \dots, k_r}(\tau) = \sum_{\substack{\lambda_1 \succ \dots \succ \lambda_r \succ 0 \\ \lambda_i \in \mathbb{Z}\tau + \mathbb{Z}}} \frac{1}{\lambda_1^{k_1} \dots \lambda_r^{k_r}}.$$

It is easy to see that these are holomorphic functions in the upper half plane and that they fulfill the **harmonic product**, i.e. it is for example

$$G_4(\tau) \cdot G_3(\tau) = G_{4,3}(\tau) + G_{3,4}(\tau) + G_7(\tau).$$

### ③ Multiple Eisenstein series - Classical Eisenstein series

In depth one we have for  $k \geq 3$

$$G_k(\tau) = \sum_{\substack{\lambda \in \mathbb{Z}\tau + \mathbb{Z} \\ \lambda \succ 0}} \frac{1}{\lambda^k} = \sum_{\substack{m > 0 \\ \vee (m=0 \wedge n > 0)}} \frac{1}{(m\tau + n)^k} = \zeta(k) + \sum_{m > 0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}$$

### ③ Multiple Eisenstein series - Classical Eisenstein series

In depth one we have for  $k \geq 3$

$$G_k(\tau) = \sum_{\substack{\lambda \in \mathbb{Z}\tau + \mathbb{Z} \\ \lambda \succ 0}} \frac{1}{\lambda^k} = \sum_{\substack{m > 0 \\ \vee (m=0 \wedge n > 0)}} \frac{1}{(m\tau + n)^k} = \zeta(k) + \sum_{m > 0} \sum_{n \in \mathbb{Z}} \frac{1}{(m\tau + n)^k}$$

By the Lipschitz summation formula we get for  $k \geq 2$  ( $q = e^{2\pi i\tau}$ )

$$\sum_{n \in \mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{d > 0} d^{k-1} q^d.$$

This gives

$$G_k(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{\substack{m > 0 \\ d > 0}} d^{k-1} q^{md} = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n.$$

### ③ Multiple Eisenstein series - Multiple version of $g$

What is a multiple version of the divisor sum?

$$g(k) = \frac{1}{(k-1)!} \sum_{\substack{m \geq 0 \\ d \geq 0}} d^{k-1} q^{md} = \frac{1}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n .$$

### ③ Multiple Eisenstein series - Multiple version of $g$ & $q$ -MZV

#### Definition

For  $k_1, \dots, k_r \geq 1$  we define the  $q$ -series  $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$  by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ d_1, \dots, d_r > 0}} \frac{d_1^{k_1-1}}{(k_1-1)!} \cdots \frac{d_r^{k_r-1}}{(k_r-1)!} q^{m_1 d_1 + \dots + m_r d_r}.$$

These  $q$ -series have a nice combinatorial interpretation

$$g(k_1, \dots, k_r) = \sum_{n > 0} \left( \quad \right) q^n.$$

### ③ Multiple Eisenstein series - Multiple version of $g$ & $q$ -MZV

#### Definition

For  $k_1, \dots, k_r \geq 1$  we define the  $q$ -series  $g(k_1, \dots, k_r) \in \mathbb{Q}[[q]]$  by

$$g(k_1, \dots, k_r) = \sum_{\substack{m_1 > \dots > m_r > 0 \\ d_1, \dots, d_r > 0}} \frac{d_1^{k_1-1}}{(k_1-1)!} \cdots \frac{d_r^{k_r-1}}{(k_r-1)!} q^{m_1 d_1 + \dots + m_r d_r}.$$

Proposition ( $g$  are  $q$ -analogues of MZVs)

For  $k_1 \geq 2, k_2, \dots, k_r \geq 1$  we have

$$\lim_{q \rightarrow 1} (1-q)^{k_1 + \dots + k_r} g(k_1, \dots, k_r) = \zeta(k_1, \dots, k_r).$$



## ③ Multiple Eisenstein series - Fourier expansion

Theorem (Gangl-Kaneko-Zagier 2006 ( $r = 2$ ), B. 2012 ( $r \geq 2$ ))

The multiple Eisenstein series  $G_{k_1, \dots, k_r}(\tau)$  have a Fourier expansion of the form

$$G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n>0} a_n q^n \quad (q = e^{2\pi i \tau})$$

and they can be written explicitly as a  $\mathcal{Z}[2\pi i]$ -linear combination of  $q$ -analogues of multiple zeta values  $g$ . In particular,  $a_n \in \mathcal{Z}[2\pi i]$ .

### Examples

$$G_k(\tau) = \zeta(k) + (-2\pi i)^k g(k),$$

$$G_{3,2}(q) = \zeta(3, 2) + 3\zeta(3)(-2\pi i)^2 g(2) + 2\zeta(2)(-2\pi i)^3 g(3) + (-2\pi i)^5 g(3, 2).$$

### ③ Multiple Eisenstein series - Do they satisfy double shuffle?

We saw the following example:

#### Example

$$\begin{aligned}\zeta(2) \cdot \zeta(3) &\stackrel{\text{harmonic}}{=} \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &\stackrel{\text{shuffle}}{=} \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1) . \\ \implies 2\zeta(3, 2) + 6\zeta(4, 1) &\stackrel{\text{double shuffle}}{=} \zeta(5) .\end{aligned}$$

#### Question

Are these relations also satisfied by the multiple Eisenstein series?

### ③ Multiple Eisenstein series - Do they satisfy double shuffle?

We saw the following example:

#### Example

$$\begin{aligned}\zeta(2) \cdot \zeta(3) &\stackrel{\text{harmonic}}{=} \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &\stackrel{\text{shuffle}}{=} \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1) . \\ \implies 2\zeta(3, 2) + 6\zeta(4, 1) &\stackrel{\text{double shuffle}}{=} \zeta(5) .\end{aligned}$$

#### Question

Are these relations also satisfied by the multiple Eisenstein series?

Problem: No definition of  $G_{2,3}$  and  $G_{4,1}$ !

### ③ Multiple Eisenstein series - Do they satisfy double shuffle?

There are different ways to extend the definition of  $G_{k_1, \dots, k_r}$  to  $k_1, \dots, k_r \geq 1$

- Formal double zeta space realization  $G_{r,s}$  (Gangl-Kaneko-Zagier, 2006)

$$\begin{aligned} G_{k_1} \cdot G_{k_2} + (\delta_{k_1,2} + \delta_{k_2,2}) \frac{G'_{k_1+k_2-2}}{2(k_1+k_2-2)} &= G_{k_1,k_2} + G_{k_2,k_1} + G_{k_1+k_2} \\ &= \sum_{j=2}^{k_1+k_2-1} \left( \binom{j-1}{k_1-1} + \binom{j-1}{k_2-1} \right) G_{j,k_1+k_2-j}, \quad (k_1+k_2 \geq 3). \end{aligned}$$

- Finite double shuffle version  $G_{r,s}$  (Kaneko, 2007).
- Shuffle regularized multiple Eisenstein series  $G_{k_1, \dots, k_r}^{\square}$  (B.-Tasaka, 2017).
- Harmonic regularized multiple Eisenstein series  $G_{k_1, \dots, k_r}^*$  (B., 2019).

#### Observation

- No version of these objects satisfy the double shuffle relations for all indices/weights.
- The derivative is always somewhere as an extra term.

### ③ Multiple Eisenstein series - Of course, they do not satisfy dsh...

Theorem (Gangl-Kaneko-Zagier +  $\epsilon$ )

For all  $k \geq 0$  there exists a basis of  $S_k$  given by explicit linear combinations of  $G_{\text{odd}, \text{odd}}$ .

Corollary (taking constant term)

For each cusp form there is a relation among  $\zeta(\text{odd}, \text{odd})$ .

**Example** There exist a  $c \in \mathbb{C}$  with

$$c\Delta = G_{3,9} - \frac{23825}{5197}G_{5,7} - \frac{41431}{10394}G_{7,5} + \frac{360}{5197}G_{9,3} + G_{11,1},$$

which implies the relation

$$0 = \zeta(3, 9) - \frac{23825}{5197}\zeta(5, 7) - \frac{41431}{10394}\zeta(7, 5) + \frac{360}{5197}\zeta(9, 3) + \zeta(11, 1).$$

Conjecturally these are the only relations among  $\zeta(\text{odd}, \text{odd})$

$\rightsquigarrow$  Explanation of  $O(X) O(X) - S(X)$  in the Broadhurst-Kreimer conjecture.

## ④ Extension of the DSH relations - General Idea

### Questions

- What are the relations satisfied by multiple Eisenstein series?
- Can we formalize these relation?

The double shuffle relations can also be stated in terms of **generating series**:

$$Z^*(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} \zeta^*(k_1, \dots, k_r) X_1^{k_1-1} \dots X_r^{k_r-1}$$

Then the extended double shuffle relations in lowest depths can be written as

$$\begin{aligned} Z^*(X_1)Z^*(X_2) &= Z^*(X_1, X_2) + Z^*(X_2, X_1) + \frac{Z^*(X_1) - Z^*(X_2)}{X_1 - X_2} \\ &= Z^*(X_1 + X_2, X_2) + Z^*(X_1 + X_2, X_1) + \zeta(2). \end{aligned}$$

## ④ Extension of the DSH relations - Formal double shuffle relations

- $A$ :  $\mathbb{Q}$ -algebra.
- For  $z_{k_1, \dots, k_r} \in A$  for  $k_1, \dots, k_r \geq 1$  we write

$$Z(X_1, \dots, X_r) = \sum_{k_1, \dots, k_r \geq 1} z_{k_1, \dots, k_r} X_1^{k_1-1} \dots X_r^{k_r-1}.$$

- A collection  $Z = (Z(X_1, \dots, X_r))_{r \geq 1}$  will be called a **mould**.

### Definition

A mould  $Z$  satisfies the **double shuffle relations** (in depth 2) if

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

## ④ Extension of the DSH relations - Known solutions

- $A = \mathbb{R}$ : Harmonic regularized multiple zeta values

$$z_{k_1, \dots, k_r} = \zeta^*(k_1, \dots, k_r).$$

- $A = \mathbb{Q}$ : Explicit solutions are known up to depth 3 (Brown, Ecalle, Gangl-Kaneko-Zagier, Tasaka). In depth 1 they are all given by

$$z_k = \begin{cases} -\frac{B_k}{2k!} = \frac{\zeta(k)}{(2\pi i)^k}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}.$$

- $A = \mathbb{Q}$ : Solution exist in all depths (Drinfel'd + Furusho, Racinet).



## ④ Extension of the DSH relations - General Idea

### General idea

- Include also (arbitrary) derivatives as objects.
- Instead of series  $Z(X_1, \dots, X_r)$  we will consider generating series with two types of variables  $X_i$  and  $Y_i$ .
- Roughly:  $X_i$ : weight,  $Y_i$ : derivative.
- In the case  $Y_i = 0$ , we get back our original story.

$A$ :  $\mathbb{Q}$ -algebra

$$B \begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} \in A[[X_1, Y_1, \dots, X_r, Y_r]].$$

### Definition

A collection  $B = \left( B \begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} \right)_{r \geq 1}$  will be called a **bimould**.

## ④ Extension of the DSH relations - Symmetril

### Definition

A bimould  $B$  is **symmetril** (up to depth 2), if

$$B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) B\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) = B\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1 \end{smallmatrix}\right) + \frac{B\left(\begin{smallmatrix} X_1 \\ Y_1+Y_2 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_2 \\ Y_1+Y_2 \end{smallmatrix}\right)}{X_1 - X_2}.$$

### Remark

- Can be written down explicitly in arbitrary depth.
- This corresponds to the harmonic product of MZV, i.e. compare it to

$$Z(X_1)Z(X_2) = Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2}.$$

#### ④ Extension of the DSH relations - Swap

##### Definition

A bimould  $B$  is called **swap invariant** if

$$B \begin{pmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{pmatrix} = B \begin{pmatrix} Y_1 + \dots + Y_r, Y_1 + \dots + Y_{r-1}, \dots, Y_1 + Y_2, Y_1 \\ X_r, X_{r-1} - X_r, \dots, X_2 - X_3, X_1 - X_2 \end{pmatrix}.$$

$$B \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix} \stackrel{\text{SW}}{=} B \begin{pmatrix} Y_1 \\ X_1 \end{pmatrix}, \quad B \begin{pmatrix} X_1, X_2 \\ Y_1, Y_2 \end{pmatrix} \stackrel{\text{SW}}{=} B \begin{pmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{pmatrix}.$$

## ④ Extension of the DSH relations - q-shuffle

Recall **symmetrility** and **swap** in low depth

$$B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) \stackrel{\text{sw}}{=} B\left(\begin{smallmatrix} Y_1 \\ X_1 \end{smallmatrix}\right), \quad B\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) \stackrel{\text{sw}}{=} B\left(\begin{smallmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{smallmatrix}\right),$$

$$B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) B\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) \stackrel{\text{il}}{=} B\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1 \end{smallmatrix}\right) + \frac{B\left(\begin{smallmatrix} X_1 \\ Y_1 + Y_2 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_2 \\ Y_1 + Y_2 \end{smallmatrix}\right)}{X_1 - X_2}.$$

### Definition

Swap + Symmetril + Swap = **q-shuffle**

$$B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) B\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) \stackrel{\text{sw}}{=} B\left(\begin{smallmatrix} Y_1 \\ X_1 \end{smallmatrix}\right) B\left(\begin{smallmatrix} Y_2 \\ X_2 \end{smallmatrix}\right)$$

$$\stackrel{\text{il}}{=} B\left(\begin{smallmatrix} Y_1, Y_2 \\ X_1, X_2 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} Y_2, Y_1 \\ X_2, X_1 \end{smallmatrix}\right) + \frac{B\left(\begin{smallmatrix} Y_1 \\ X_1 + X_2 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} Y_2 \\ X_1 + X_2 \end{smallmatrix}\right)}{Y_1 - Y_2}$$

$$\stackrel{\text{sw}}{=} B\left(\begin{smallmatrix} X_1 + X_2, X_1 \\ Y_2, Y_1 - Y_2 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_1 + X_2, X_2 \\ Y_1, Y_2 - Y_1 \end{smallmatrix}\right) + \frac{B\left(\begin{smallmatrix} X_1 + X_2 \\ Y_1 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_1 + X_2 \\ Y_2 \end{smallmatrix}\right)}{Y_1 - Y_2}.$$

## ④ Extension of the DSH relations - q-double shuffle

### Definition

A bimould satisfies **q-double shuffle** (in depth 2) if

$$\begin{aligned} B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) B\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) &= B\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1 \end{smallmatrix}\right) + \frac{B\left(\begin{smallmatrix} X_1 \\ Y_1+Y_2 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_2 \\ Y_1+Y_2 \end{smallmatrix}\right)}{X_1 - X_2} \\ &= B\left(\begin{smallmatrix} X_1 + X_2, X_1 \\ Y_2, Y_1 - Y_2 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_1 + X_2, X_2 \\ Y_1, Y_2 - Y_1 \end{smallmatrix}\right) + \frac{B\left(\begin{smallmatrix} X_1+X_2 \\ Y_1 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_1+X_2 \\ Y_2 \end{smallmatrix}\right)}{Y_1 - Y_2}, \end{aligned}$$

i.e.  $B$  is symmetril and satisfies the  $q$ -shuffle product formula.

- Clearly: Symmetril + Swap invariant  $\implies$  q-double shuffle.
- Compare this to the double shuffle relations

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

#### ④ Extension of the DSH relations - "Constant function" $\rightsquigarrow$ Sol. to dsh

Solution to q-dsh  $\Rightarrow$  solution to dsh

##### Proposition

Let  $B$  be **symmetril** and **swap invariant** with  $\frac{d}{dX} \frac{d}{dY} B \begin{pmatrix} X \\ Y \end{pmatrix} = 0$ . Then

$$Z(X) = B \begin{pmatrix} X \\ 0 \end{pmatrix}, \quad Z(X_1, X_2) = B \begin{pmatrix} X_1, X_2 \\ 0, 0 \end{pmatrix}$$

satisfies the double shuffle relations.

**Proof:**

$$\begin{aligned} \frac{B \begin{pmatrix} X_1 + X_2 \\ Y_1 \end{pmatrix} - B \begin{pmatrix} X_1 + X_2 \\ Y_2 \end{pmatrix}}{Y_1 - Y_2} \Big|_{Y_1 = Y_2 = 0} &= \sum_{k \geq 1} b \binom{k}{1} (X_1 + X_2)^{k-1} \\ &= b \binom{1}{1} \stackrel{\text{SW}}{=} b \binom{2}{0} = z_2. \end{aligned}$$

**Interpretation:** "When the derivative vanishes (constant function) then we obtain a solution to classical dsh (equations for numbers)".

#### ④ Extension of the DSH relations - Sol. to dsh $\rightsquigarrow$ Sol. to $q$ -dsh

Theorem (B.-Kühn-Matthes, 2020+)

Let  $Z$  satisfy the double shuffle relations (in all depths), then there exists an explicit construction of a symmetril and swap invariant bimould  $B$ .

For example, in lowest depth the bimould

$$B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) = Z(X_1) + Z(Y_1),$$
$$B\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) = Z(X_1, X_2) + Z(Y_1 + Y_2, Y_1) + Z(X_2)Z(Y_1) + \frac{1}{2}z_2$$

is **symmetril** and **swap invariant**

**Interpretation:** "Numbers can be viewed as constant functions".

## ④ Extension of the DSH relations - Combinatorial MES

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

There exist a **symmetril** and **swap invariant** bimould  $\mathfrak{G}$  (up to depth 3) which in depth one is given by the generating series of derivatives of Eisenstein series.

### Remark

- This setup gives combinatorial proofs of classical identities (e.g. Ramanujan Diff.eq.).
- The construction of this bimould is inspired by the Fourier expansion of multiple Eisenstein series.
- The bimould  $\mathfrak{G}$  can be written down explicitly in terms of rational solutions to the classical double shuffle equations and a bi-variant of the  $q$ -series  $g$ .
- We have a conjectured construction for all depths (j.w. A. Burmester).



## ④ Extension of the DSH relations - Combinatorial MES

### Definition

We define the **combinatorial multiple Eisenstein series**  $G$  in depth  $\leq 2$  by

$$\mathfrak{G}\left(\begin{matrix} X \\ Y \end{matrix}\right) =: \sum_{\substack{k \geq 1 \\ d \geq 0}} G\left(\begin{matrix} k \\ d \end{matrix}\right) X^{k-1} \frac{Y^d}{d!},$$

$$\mathfrak{G}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) =: \sum_{\substack{k_1, k_2 \geq 1 \\ d_1, d_2 \geq 0}} G\left(\begin{matrix} k_1, k_2 \\ d_1, d_2 \end{matrix}\right) X_1^{k_1-1} X_2^{k_2-1} \frac{Y_1^{d_1}}{d_1!} \frac{Y_2^{d_2}}{d_2!}.$$

- In depth one  $G\left(\begin{smallmatrix} k \\ d \end{smallmatrix}\right)$  is basically the  $d$ -th derivative of  $G_{k-d}$ .
- In depth two the  $G\left(\begin{smallmatrix} k_1, k_2 \\ 0, 0 \end{smallmatrix}\right)$  are (almost) the double Eisenstein series.
- The symmetry  $\mathfrak{G}$  of gives

$$G\left(\begin{matrix} k_1 \\ d_1 \end{matrix}\right) G\left(\begin{matrix} k_2 \\ d_2 \end{matrix}\right) = G\left(\begin{matrix} k_1, k_2 \\ d_1, d_2 \end{matrix}\right) + G\left(\begin{matrix} k_2, k_1 \\ d_2, d_1 \end{matrix}\right) + G\left(\begin{matrix} k_1 + k_2 \\ d_1 + d_2 \end{matrix}\right).$$

## ④ Extension of the DSH relations - The space of CMES

### Definition

Space of double combinatorial multiple Eisenstein series of weight  $K \geq 1$ :

$$\mathfrak{D}_K = \left\langle G\binom{k}{d}, G\binom{k_1, k_2}{d_1, d_2} \mid \begin{array}{l} k+d=k_1+k_2+d_1+d_2=K \\ k, k_1, k_2 \geq 1, d, d_1, d_2 \geq 0 \end{array} \right\rangle_{\mathbb{Q}}$$

### Proposition

$$\begin{aligned} q \frac{d}{dq} \mathfrak{G} \left( \begin{array}{c} X_1 \\ Y_1 \end{array} \right) &= \frac{d}{dX_1} \frac{d}{dY_1} \mathfrak{G} \left( \begin{array}{c} X_1 \\ Y_1 \end{array} \right), \\ q \frac{d}{dq} \mathfrak{G} \left( \begin{array}{c} X_1, X_2 \\ Y_1, Y_2 \end{array} \right) &= \left( \frac{d}{dX_1} \frac{d}{dY_1} + \frac{d}{dX_2} \frac{d}{dY_2} \right) \mathfrak{G} \left( \begin{array}{c} X_1, X_2 \\ Y_1, Y_2 \end{array} \right). \end{aligned}$$

### Corollary

Combinatorial multiple Eisenstein series are closed under  $q \frac{d}{dq}$ . In particular

$$q \frac{d}{dq} \mathfrak{D}_K \subset \mathfrak{D}_{K+2}.$$

#### ④ Extension of the DSH relations - The space of CMES

$$\mathfrak{D}_K = \left\langle G\binom{k}{d}, G\binom{k_1, k_2}{d_1, d_2} \mid \begin{array}{l} k+d=k_1+k_2+d_1+d_2=K \\ k, k_1, k_2 \geq 1, d, d_1, d_2 \geq 0 \end{array} \right\rangle_{\mathbb{Q}}$$

$$\mathfrak{D}_K^0 = \left\langle G\binom{k}{0}, G\binom{k_1, k_2}{0, 0} \in \mathfrak{D}_K \right\rangle_{\mathbb{Q}}$$

##### Proposition

- $\mathfrak{D}_K$  contains the space of **quasi modular forms**  $\mathbb{Q}[\tilde{G}_2, \tilde{G}_4, \tilde{G}_6]_K$  of weight  $K$ .
- $\mathfrak{D}_K^0$  contains the space of **modular forms**  $\mathbb{Q}[\tilde{G}_4, \tilde{G}_6]_K$  of weight  $K$

Numerical computer calculation give:

$k$	1	2	3	4	5	6	7	8
$\dim \mathfrak{D}_K \stackrel{?}{=}$	1	2	3	5	7	11	14	..
$\dim \mathfrak{D}_K^0 \stackrel{?}{=}$	1	2	3	3	4	4	5	5
# generators of $\mathfrak{D}_K$	1	3	7	14	25	41	63	92

# Numbers

# Functions / q-series

“single”  
objects

Riemann zeta values

$$\zeta(k)$$

Eisenstein series

$$G_k(\tau) = \zeta(k) + \frac{(2\pi i)^k}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n}$$

$$(q = e^{2\pi i \tau})$$

“multiple”  
objects

Multiple zeta values

$$\zeta(k_1, \dots, k_r)$$

Multiple Eisenstein series

$$G_{k_1, \dots, k_r}(\tau) = \zeta(k_1, \dots, k_r) + \sum_{n > 0} a_n q^n$$

relations

Double shuffle relations

$$\begin{aligned} \zeta(2) \cdot \zeta(3) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(5) \\ &= \zeta(2, 3) + 3\zeta(3, 2) + 6\zeta(4, 1) \end{aligned}$$

Symmetril & Swap invariant

object point  
of view

generating  
series point  
of view

$$\begin{aligned} Z(X_1)Z(X_2) &= Z(X_1, X_2) + Z(X_2, X_1) + \frac{Z(X_1) - Z(X_2)}{X_1 - X_2} \\ &= Z(X_1 + X_2, X_1) + Z(X_1 + X_2, X_2) + z_2. \end{aligned}$$

$$\begin{aligned} B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) &\stackrel{\text{sw}}{=} B\left(\begin{smallmatrix} Y_1 \\ X_1 \end{smallmatrix}\right), \quad B\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) \stackrel{\text{sw}}{=} B\left(\begin{smallmatrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{smallmatrix}\right), \\ B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) B\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) &\stackrel{\text{sw}}{=} B\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1 \end{smallmatrix}\right) + \frac{B\left(\begin{smallmatrix} X_1 \\ Y_1 + Y_2 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_2 \\ Y_1 + Y_2 \end{smallmatrix}\right)}{X_1 - X_2}. \end{aligned}$$

## ⑤ Bonus - Basis conjecture

### Conjecture (Hoffman, 1997)

For  $k \geq 0$  the multiple zeta values

$$\{\zeta(k_1, \dots, k_r) \mid r \geq 0, k_1 + \dots + k_r = k, k_1, \dots, k_r \in \{2, 3\}\}$$

form a basis of  $\mathcal{Z}_k$ .

$$\zeta(2n) \in \pi^{2n}\mathbb{Q}, \quad \zeta(2, \dots, 2) = \frac{\pi^{2n}}{(2n+1)!}, \quad \zeta(5) = \frac{6}{5}\zeta(2, 3) + \frac{4}{5}\zeta(3, 2).$$

### Theorem (Brown, 2012)

For all  $k \geq 0$  we have

$$\mathcal{Z}_k = \langle \zeta(k_1, \dots, k_r) \mid r \geq 0, k_1 + \dots + k_r = k, k_1, \dots, k_r \in \{2, 3\} \rangle_{\mathbb{Q}}.$$

## ⑤ Bonus - Symmetril in depth 3

### Definition

A bimould  $B$  is **symmetril** (up to depth 3), if

$$B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) B\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) = B\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1 \end{smallmatrix}\right) + \frac{B\left(\begin{smallmatrix} X_1 \\ Y_1+Y_2 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_2 \\ Y_1+Y_2 \end{smallmatrix}\right)}{X_1 - X_2},$$

$$\begin{aligned} B\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) B\left(\begin{smallmatrix} X_2, X_3 \\ Y_2, Y_3 \end{smallmatrix}\right) &= B\left(\begin{smallmatrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_2, X_1, X_3 \\ Y_2, Y_1, Y_3 \end{smallmatrix}\right) + B\left(\begin{smallmatrix} X_2, X_3, X_1 \\ Y_2, Y_3, Y_1 \end{smallmatrix}\right) \\ &+ \frac{B\left(\begin{smallmatrix} X_1, X_3 \\ Y_1+Y_2, Y_3 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_2, X_3 \\ Y_1+Y_2, Y_3 \end{smallmatrix}\right)}{X_1 - X_2} + \frac{B\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1+Y_3 \end{smallmatrix}\right) - B\left(\begin{smallmatrix} X_2, X_3 \\ Y_2, Y_1+Y_3 \end{smallmatrix}\right)}{X_1 - X_3}. \end{aligned}$$

## ⑤ Bonus - The bimould $\mathfrak{g}$

Define for  $m \geq 1$  the series

$$L_m \left( \begin{smallmatrix} X \\ Y \end{smallmatrix} \right) = \frac{e^{X+mY} q^m}{1 - e^X q^m} = \sum_{n \geq 1} e^{nX+mY} q^{mn}.$$

### Definition

We define the bimould  $\mathfrak{g}$  for all depth  $r \geq 1$  by

$$\mathfrak{g} \left( \begin{smallmatrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{smallmatrix} \right) = \sum_{m_1 > \dots > m_r > 0} L_{m_1} \left( \begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix} \right) \dots L_{m_r} \left( \begin{smallmatrix} X_r \\ Y_r \end{smallmatrix} \right).$$

### Proposition

The bimould  $\mathfrak{g}$  is **swap invariant**.

## ⑤ Bonus - The bimould $\mathfrak{g}$

**Example:** Swap invariance of  $\mathfrak{g}$  in depth 2

$$\begin{aligned}\mathfrak{g}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) &= \sum_{m_1 > m_2 > 0} L_{m_1}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_2 \\ Y_2 \end{matrix}\right) \\ &= \sum_{\substack{m_1 > m_2 > 0 \\ n_1, n_2 > 0}} e^{n_1 X_1 + n_2 X_2 + m_1 Y_1 + m_2 Y_2} q^{m_1 n_1 + m_2 n_2} = (\star)\end{aligned}$$

**Change of variables**  $\longleftrightarrow$  **swap of variables**

$$\left\{ \begin{array}{l} m_1 = m'_1 + m'_2 \\ n_1 = n'_2 \end{array} \right., \left. \begin{array}{l} m_2 = m'_1 \\ n_2 = n'_1 - n'_2 \end{array} \right\} \implies m_1 n_1 + m_2 n_2 = m'_1 n'_1 + m'_2 n'_2.$$

$$\begin{aligned}(\star) &= \sum_{\substack{m'_1, m'_2 > 0 \\ n'_1 > n'_2 > 0}} e^{n'_2 X_1 + (n'_1 - n'_2) X_2 + (m'_1 + m'_2) Y_1 + m'_1 Y_2} q^{m'_1 n'_1 + m'_2 n'_2} \\ &= \mathfrak{g}\left(\begin{matrix} Y_1 + Y_2, Y_1 \\ X_2, X_1 - X_2 \end{matrix}\right).\end{aligned}$$



## ⑤ Bonus - Product of $\mathfrak{g}$

$$\begin{aligned}\mathfrak{g}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right)\mathfrak{g}\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) &= \sum_{m_1>0} L_{m_1}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) \sum_{m_2>0} L_{m_2}\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) \\ &= \left( \sum_{m_1>m_2>0} + \sum_{m_2>m_1>0} + \sum_{m_1=m_2>0} \right) L_{m_1}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) L_{m_2}\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) \\ &= \mathfrak{g}\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + \mathfrak{g}\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1 \end{smallmatrix}\right) + \sum_{m>0} L_m\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) L_m\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right).\end{aligned}$$

To describe the product of  $\mathfrak{g}$  we need to describe for a fixed  $m$  the product of  $L_m$ .

## ⑤ Bonus - Product of $\mathfrak{g}$

$$\begin{aligned}
 \mathfrak{g}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) \mathfrak{g}\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) &= \sum_{m_1 > 0} L_{m_1}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) \sum_{m_2 > 0} L_{m_2}\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) \\
 &= \left( \sum_{m_1 > m_2 > 0} + \sum_{m_2 > m_1 > 0} + \sum_{m_1 = m_2 > 0} \right) L_{m_1}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) L_{m_2}\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) \\
 &= \mathfrak{g}\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + \mathfrak{g}\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1 \end{smallmatrix}\right) + \sum_{m > 0} L_m\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) L_m\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right).
 \end{aligned}$$

### Lemma

For all  $m \geq 1$  we have

$$L_m\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) L_m\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right) = \frac{L_m\left(\begin{smallmatrix} X_1 \\ Y_1 + Y_2 \end{smallmatrix}\right) - L_m\left(\begin{smallmatrix} X_2 \\ Y_1 + Y_2 \end{smallmatrix}\right)}{X_1 - X_2} + L_m\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + L_m\left(\begin{smallmatrix} X_2, X_1 \\ Y_2, Y_1 \end{smallmatrix}\right)$$

where

$$L_m\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) = L_m\left(\begin{smallmatrix} X_1 \\ Y_1 + Y_2 \end{smallmatrix}\right) \left( \beta\left(\begin{smallmatrix} X_2 - X_1 \\ -Y_2 \end{smallmatrix}\right) - \frac{1}{2} \right) + \beta\left(\begin{smallmatrix} X_1 - X_2 \\ Y_1 \end{smallmatrix}\right) L_m\left(\begin{smallmatrix} X_2 \\ Y_1 + Y_2 \end{smallmatrix}\right)$$

## ⑤ Bonus - Combinatorial MES

### Proposition

For all  $m \geq 1$  the series

$$L_m \left( \begin{matrix} X \\ Y \end{matrix} \right) = \frac{e^{X+mY} q^m}{1 - e^X q^m},$$

$$L_m \left( \begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix} \right) = L_m \left( \begin{matrix} X_1 \\ Y_1 + Y_2 \end{matrix} \right) \left( \beta \left( \begin{matrix} X_2 - X_1 \\ -Y_2 \end{matrix} \right) - \frac{1}{2} \right) + \beta \left( \begin{matrix} X_1 - X_2 \\ Y_1 \end{matrix} \right) L_m \left( \begin{matrix} X_2 \\ Y_1 + Y_2 \end{matrix} \right),$$

$$L_m \left( \begin{matrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{matrix} \right) = \text{explicit long formula}$$

are **symmetril**.

### Remark

- The  $L_m \left( \begin{matrix} X \\ Y \end{matrix} \right)$  can be seen as the the generating series of "bi-monotangent" function.
- The construction of  $L_m \left( \begin{matrix} X_1, \dots, X_r \\ Y_1, \dots, Y_r \end{matrix} \right)$  in terms of  $\beta$  and  $L_m \left( \begin{matrix} X \\ Y \end{matrix} \right)$  corresponds to "Multitangent = MZV-linear combination of monotangent".

## ⑤ Bonus - Make $\mathfrak{g}$ symmetril

### Proposition

If  $L_m$  is **symmetril** for all  $m \geq 1$ , then

$$\begin{aligned}\mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) &= \sum_{m>0} L_m\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right), \\ \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) &= \sum_{m_1>m_2>0} L_{m_1}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_2 \\ Y_2 \end{matrix}\right) + \sum_{m>0} L_m\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right), \\ \mathfrak{g}^{\text{il}}\left(\begin{matrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{matrix}\right) &= \sum_{m_1>m_2>m_3>0} L_{m_1}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_2 \\ Y_2 \end{matrix}\right) L_{m_3}\left(\begin{matrix} X_3 \\ Y_3 \end{matrix}\right) \\ &\quad + \sum_{m_1>m_2>0} \left( L_{m_1}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_3 \\ Y_3 \end{matrix}\right) + L_{m_1}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) L_{m_2}\left(\begin{matrix} X_2, X_3 \\ Y_2, Y_3 \end{matrix}\right) \right) \\ &\quad + \sum_{m>0} L_m\left(\begin{matrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{matrix}\right),\end{aligned}$$

are also **symmetril**.

## ⑤ Bonus - Combinatorial MES

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

The following series are **symmetril** and **swap invariant**

$$\mathfrak{G}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) = \mathfrak{g}^{\text{il}}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) + \beta\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right),$$

$$\mathfrak{G}\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) = \mathfrak{g}^{\text{il}}\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right) + \mathfrak{g}^{\text{il}}\left(\begin{smallmatrix} X_2 \\ Y_2 \end{smallmatrix}\right)\beta\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right) + \beta\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right),$$

$$\begin{aligned}\mathfrak{G}\left(\begin{smallmatrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{smallmatrix}\right) &= \mathfrak{g}^{\text{il}}\left(\begin{smallmatrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{smallmatrix}\right) + \mathfrak{g}^{\text{il}}\left(\begin{smallmatrix} X_1, X_2 \\ Y_1, Y_2 \end{smallmatrix}\right)\beta\left(\begin{smallmatrix} X_3 \\ Y_3 \end{smallmatrix}\right) \\ &\quad + \mathfrak{g}^{\text{il}}\left(\begin{smallmatrix} X_1 \\ Y_1 \end{smallmatrix}\right)\beta\left(\begin{smallmatrix} X_2, X_3 \\ Y_2, Y_3 \end{smallmatrix}\right) + \beta\left(\begin{smallmatrix} X_1, X_2, X_3 \\ Y_1, Y_2, Y_3 \end{smallmatrix}\right).\end{aligned}$$

In the mould language:  $\mathfrak{G}$  is the mould product of the two symmetril bimoulds  $\mathfrak{g}^{\text{il}}$  and  $\beta$ .

## ⑤ Bonus - Combinatorial MES explicit

Theorem (B.-Kühn-Matthes 2020+, B.-Burmester 2020+)

The following series are **symmetril** and **swap invariant**

$$\begin{aligned}\mathfrak{G}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) &= \beta\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) + \mathfrak{g}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right), \\ \mathfrak{G}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) &= \beta\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right) - \beta\left(\begin{matrix} X_1 - X_2 \\ Y_2 \end{matrix}\right) \mathfrak{g}\left(\begin{matrix} X_1 \\ Y_1 + Y_2 \end{matrix}\right) - \frac{1}{2} \mathfrak{g}\left(\begin{matrix} X_1 \\ Y_1 + Y_2 \end{matrix}\right) \\ &\quad + \beta\left(\begin{matrix} X_2 \\ Y_2 \end{matrix}\right) \mathfrak{g}\left(\begin{matrix} X_1 \\ Y_1 \end{matrix}\right) + \beta\left(\begin{matrix} X_1 - X_2 \\ Y_1 \end{matrix}\right) \mathfrak{g}\left(\begin{matrix} X_2 \\ Y_1 + Y_2 \end{matrix}\right) + \mathfrak{g}\left(\begin{matrix} X_1, X_2 \\ Y_1, Y_2 \end{matrix}\right).\end{aligned}$$

- In the mould language:  $\mathfrak{G}$  is the mould product of the two symmetril bimoulds  $\mathfrak{g}^{\text{il}}$  and  $\beta$ .
- For the construction for depth  $\geq 3$  (which conjecturally works in all depths) see the talkslides of Annika Burmesters talk "Combinatorial multiple Eisenstein series" at the JENTE Seminar (<https://sites.google.com/view/jente-seminar/home>).