Derivatives of $q$-analogues of multiple zeta values

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Slides are available here: www.henrikbachmann.com
Content of this talk

- Overview & Multiple zeta values
- Different models of $q$-analogues of multiple zeta values
- Harmonic & Shuffle product
- Shuffle product $\leftrightarrow$ Derivatives
- Certain interesting subspaces and their derivatives
Multiple zeta values

Definition

For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the **multiple zeta value** (MZV) by

$$
\zeta(k_1, \ldots, k_r) = \sum_{0 < m_1 < \cdots < m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.
$$

By $r$ we denote its **depth** and $k_1 + \cdots + k_r$ will be called its **weight**. For the $\mathbb{Q}$-vector space spanned by all multiple zeta values we write $\mathcal{Z}$.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (**harmonic product**). e.g:

$$
\zeta(k_1) \cdot \zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).
$$

- MZV can be expressed as iterated integrals. This gives another way (**shuffle product**) to express the product of two MZV as a linear combination of MZV.

- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV.
"Roughly speaking, in mathematics, specifically in the areas of combinatorics and special functions, a $q$-analogue of a theorem, identity or expression is a generalization involving a new parameter $q$ that returns the original theorem, identity or expression in the limit as $q \to 1$."

- The easiest example is the $q$-analogue of a natural number $m$ given by
  \[
  [m]_q = \frac{1 - q^m}{1 - q} = 1 + q + \cdots + q^{m-1}, \quad \lim_{q \to 1} [m]_q = m.
  \]

- One approach to get an $q$-analogue of multiple zeta values is to replace $\frac{1}{m^k}$ by $\frac{q^{(k-1)m}}{[m]_q^k}$. 
**Definition (Bradley, Zhao)**

For $k_1, \ldots, k_{r-1} \geq 1$, $k_r \geq 2$ define the $q$-multiple zeta value by

$$
\zeta_q(k_1, \ldots, k_r) = \sum_{0 < m_1 < \cdots < m_r} \frac{q^{(k_1-1)m_1} \cdots q^{(k_r-1)m_r}}{[m_1]^q_{k_1} \cdots [m_r]^q_{k_r}}.
$$

- Clearly it is $\lim_{q \to 1} \zeta_q(k_1, \ldots, k_r) = \zeta(k_1, \ldots, k_r)$.
- The $\mathbb{Q}$-vector space spanned by these series form a $\mathbb{Q}[q]$-algebra.

$$
\zeta_q(2)\zeta_q(3) = \zeta_q(2, 3) + \zeta_q(3, 2) + \zeta_q(5) + (1 - q)\zeta_q(4).
$$
The algebraic description of $q$-analogues become easier by removing the factors $(1 - q)^k$.

**Definition (Okuda-Takeyama)**

- For $k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2$ define the (modified) $q$-multiple zeta value by

$$
\zeta_q(k_1, \ldots, k_r) = \sum_{0 < m_1 < \cdots < m_r} \frac{q^{(k_1-1)m_1} \cdots q^{(k_r-1)m_r}}{(1 - q^{m_1})^{k_1} \cdots (1 - q^{m_r})^{k_r}}
$$

$$
= (1 - q)^{-k} \zeta_q(k_1, \ldots, k_r) \in \mathbb{Q}[[q]].
$$

with $k = k_1 + \cdots + k_r$.

- For the $\mathbb{Q}$-vector space spanned by these series we write

$$
\mathcal{Z}_q := \langle \zeta_q(k_1, \ldots, k_r) \mid r \geq 0, k_1, \ldots, k_{r-1} \geq 1, k_r \geq 2 \rangle_{\mathbb{Q}},
$$

and set $\zeta_q(k_1, \ldots, k_r) = 1$ for $r = 0$.

- Clearly it is $\lim_{q \to 1} (1 - q)^k \zeta_q(k_1, \ldots, k_r) = \zeta(k_1, \ldots, k_r)$.

- The $\mathbb{Q}$-vector space $\mathcal{Z}_q$ is a $\mathbb{Q}$-algebra.
More generally: Given a family of polynomials $Q_k(X)$ for $k \geq 1$ with $Q_k(1) = 1$ one can define a $q$-analogue of multiple zeta values by

$$\sum_{0 < m_1 < \cdots < m_r} \frac{Q_{k_1}(q^{m_1}) \cdots Q_{k_r}(q^{m_r})}{(1 - q^{m_1})^{k_1} \cdots (1 - q^{m_r})^{k_r}}$$

- In the case $\zeta_q$ the polynomials $Q_k(X) = X^{k-1}$ are used.
- To study the connection to modular forms the following polynomials are more useful

$$\frac{Q_k(X)}{(1 - X)^k} := \sum_{d > 0} \frac{d^{k-1}}{(k-1)!} X^d$$

- In special cases (all $k_j \geq 2$) both models are basically the same.
For $k_1, \ldots, k_r \geq 1$ we define the following $q$-series in $\mathbb{Q}[[q]]$

$$g_{k_1, \ldots, k_r}(q) := \sum_{0 < u_1 < \cdots < u_r} \frac{v_1^{k_1-1} \cdots v_r^{k_r-1}}{(k_1-1)! \cdots (k_r-1)!} \cdot q^{u_1 v_1 + \cdots + u_r v_r}.$$

By $k_1 + \cdots + k_r$ we denote its weight and by $r$ its depth.

For the $\mathbb{Q}$-vector space spanned by these series we write

$$\mathcal{G} := \left\langle g_{k_1, \ldots, k_r}(q) \mid r \geq 0, k_1, \ldots, k_r \geq 1 \right\rangle_{\mathbb{Q}},$$

where we also set $g_{k_1, \ldots, k_r}(q) = 1$ for $r = 0.$
Definition

- For $k_1, \ldots, k_r \geq 1$ we define the following $q$-series in $\mathbb{Q}[[q]]$

$$g_{k_1, \ldots, k_r}(q) := \sum_{0 < u_1 < \cdots < u_r, 0 < v_1, \ldots, v_r} \frac{v_1^{k_1-1} \cdots v_i^{k_i-1}}{(k_1 - 1)! \cdots (k_r - 1)!} \cdot q^{u_1 v_1 + \cdots + u_r v_r}.$$ 

By $k_1 + \cdots + k_r$ we denote its weight and by $r$ its depth.

- For the $\mathbb{Q}$-vector space spanned by these series we write

$$\mathcal{G} := \langle g_{k_1, \ldots, k_r}(q) \mid r \geq 0, k_1, \ldots, k_r \geq 1 \rangle_{\mathbb{Q}},$$

where we also set $g_{k_1, \ldots, k_r}(q) = 1$ for $r = 0$.

In depth one these are just the generating series of the divisor-sum $\sigma_{k-1}(n) = \sum_{d \mid n} d^{k-1}$:

$$g_k(q) = \sum_{0 < u_1 \atop 0 < v_1} \frac{v_1^{k-1}}{(k - 1)!} q^{u_1 v_1} = \frac{1}{(k - 1)!} \sum_{n > 0} \sigma_{k-1}(n) q^n.$$
For the generating function of the $q$-series $g_{k_1, \ldots, k_r}$ we write

$$g(x_1, \ldots, x_r) := \sum_{k_1, \ldots, k_r \geq 1} g_{k_1, \ldots, k_r}(q) x_1^{k_1-1} \ldots x_r^{k_r-1}.$$ 

**Lemma**

The series $g$ can be written as

$$g(x_1, \ldots, x_r) = \sum_{0 < u_1 < \cdots < u_r} L_{u_1}(x_1) \ldots L_{u_r}(x_r),$$

where

$$L_u(x) = \frac{q^u e^x}{1 - q^u e^x}.$$ 

The series $L_u(x)$ satisfy the equation

$$L_u(x) \cdot L_u(y) = \frac{L_u(x) - L_u(y)}{x - y} + B(x - y)L_u(x) + B(y - x)L_u(y)$$

with $B(T) = \sum_{k=1}^{\infty} \frac{B_k}{k!} T^{k-1}$. 
$q$-analogues of multiple zeta values: harmonic product

**MZV:** Recall that the generating series of the harmonic multiple zeta values

$$\mathcal{T}^\ast(x_1, \ldots, x_r) = \sum_{k_1, \ldots, k_r \geq 1} \zeta^\ast(k_1, \ldots, k_r) x_1^{k_1-1} \cdots x_r^{k_r-1}$$

satisfy

$$\mathcal{T}^\ast(x) \cdot \mathcal{T}^\ast(y) = \mathcal{T}^\ast(x, y) + \mathcal{T}^\ast(y, x) + \frac{\mathcal{T}^\ast(x) - \mathcal{T}^\ast(y)}{x - y}.$$ 

**qMZV:** The generating series $g(x_1, \ldots, x_r)$ satisfies similar equations, e.g.

$$g(x) \cdot g(y) = g(x, y) + g(y, x) + \frac{g(x) - g(y)}{x - y}$$

$$+ B(x - y) \cdot g(x) + B(y - x) \cdot g(y),$$

with $B(T) = \sum_{k=1}^{\infty} \frac{B_k}{k!} T^{k-1}$. 

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$q$-analogues of multiple zeta values: harmonic & shuffle product

Let $\mathbb{I}_{k_1}$ and $\mathbb{I}_{k_2}$ be two arbitrary index sets.

We have just seen the following rough picture:

**Harmonic product $\ast$**

\[
\zeta^\ast(\mathbb{I}_{k_1}) \cdot \zeta^\ast(\mathbb{I}_{k_2}) = \zeta^\ast(\mathbb{I}_{k_1 \ast \mathbb{I}_{k_2}})
\]

\[
g_{\mathbb{I}_{k_1}}(q) \cdot g_{\mathbb{I}_{k_2}}(q) = g_{\mathbb{I}_{k_1 \ast \mathbb{I}_{k_2}}}(q) + \text{lower weight terms}.
\]

Now we want to explain that something similar is true for the shuffle product:

**Shuffle product $\shuffle$**

\[
\zeta^{\shuffle}(\mathbb{I}_{k_1}) \cdot \zeta^{\shuffle}(\mathbb{I}_{k_2}) = \zeta^{\shuffle}(\mathbb{I}_{k_1 \shuffle \mathbb{I}_{k_2}})
\]

\[
g_{\mathbb{I}_{k_1}}(q) \cdot g_{\mathbb{I}_{k_2}}(q) = g_{\mathbb{I}_{k_1 \shuffle \mathbb{I}_{k_2}}}(q) + \text{lower weight terms} + \text{"derivatives"}.
\]
Let $I_{k_1}$ and $I_{k_2}$ be two arbitrary index sets.

We have just seen the following rough picture:

**Harmonic product $*$**

\[
\tilde{\zeta}^* (I_{k_1}) \cdot \tilde{\zeta}^* (I_{k_2}) = \tilde{\zeta}^* (I_{k_1} * I_{k_2}) \\
g_{I_{k_1}}(q) \cdot g_{I_{k_2}}(q) = g_{I_{k_1} \star I_{k_2}}(q) + \text{lower weight terms}.
\]

Now we want to explain that something similar is true for the shuffle product:

**Shuffle product $\sqcup$**

\[
\tilde{\zeta}^{\sqcup} (I_{k_1}) \cdot \tilde{\zeta}^{\sqcup} (I_{k_2}) = \tilde{\zeta}^{\sqcup} (I_{k_1} \sqcup I_{k_2}) \\
g_{I_{k_1}}(q) \cdot g_{I_{k_2}}(q) = g_{I_{k_1} \sqcup I_{k_2}}(q) + \text{lower weight terms} + \text{"derivatives"}.
\]

"derivatives" = $g_{I_{k_1}}(q) \cdot g_{I_{k_2}}(q) - g_{I_{k_1} \sqcup I_{k_2}}(q) - \text{lower weight terms}.$
The operator $D$

Today we will be interested in the operator $D$ on $\mathbb{Q}[[q]]$ defined by

$$D := q \frac{d}{dq}.$$ 

- For a $q$-analogue the operator $D$ increases the **weight by 2** and the **depth by 1**.
- Given numbers $k_1, \ldots, k_r \geq 1$ with $k = k_1 + \cdots + k_r$ it is

$$\lim_{q \to 1} (1 - q)^{k+2} D g_{k_1,\ldots,k_r}(q) = 0,$$

i.e. **formulas for the derivative of $q$-analogues give relations between MZV**.
- The sub algebra $\mathbb{Q}[\tilde{G}_2, \tilde{G}_4, \tilde{G}_6] \subset \mathcal{G}$ of **quasi-modular forms** is closed under $D$.

$$\tilde{G}_{2n}(q) := \frac{1}{2} \frac{B_{2n}}{(2n)!} + g_{2n}(q) \in \mathcal{G}.$$
**Definition**

Define for $n_1, \ldots, n_r \geq 1$ the series

$$H\left(n_1, \ldots, n_r \atop x_1, \ldots, x_r\right) = \sum_{0<d_1<\cdots<d_r} e^{d_1x_1}\left(\frac{q^{d_1}}{1 - q^{d_1}}\right)^{n_1} \cdots e^{d_rx_r}\left(\frac{q^{d_r}}{1 - q^{d_r}}\right)^{n_r}.$$ 

Notice that this series "satisfies" the harmonic product formula. For example:

$$H\left(n_1 \atop x_1\right) \cdot H\left(n_2 \atop x_2\right) = H\left(n_1, n_2 \atop x_1, x_2\right) + H\left(n_2, n_1 \atop x_2, x_1\right) + H\left(n_1 + n_2 \atop x_1 + x_2\right).$$
Define for \( n_1, \ldots, n_r \geq 1 \) the series

\[
H\left( \begin{array}{c}
(n_1, \ldots, n_r) \\
x_1, \ldots, x_r
\end{array} \right) = \sum_{0 < d_1 < \cdots < d_r} e^{d_1 x_1} \left( \frac{q^{d_1}}{1 - q^{d_1}} \right)^{n_1} \cdots e^{d_r x_r} \left( \frac{q^{d_r}}{1 - q^{d_r}} \right)^{n_r}.
\]

Notice that this series "satisfies" the harmonic product formula. For example:

\[
H\left( \begin{array}{c}
(n_1) \\
x_1
\end{array} \right) \cdot H\left( \begin{array}{c}
(n_2) \\
x_2
\end{array} \right) = H\left( \begin{array}{c}
(n_1, n_2) \\
x_1, x_2
\end{array} \right) + H\left( \begin{array}{c}
(n_2, n_1) \\
x_2, x_1
\end{array} \right) + H\left( \begin{array}{c}
(n_1 + n_2) \\
x_1 + x_2
\end{array} \right).
\]

The connection to the series \( g \) is given by

\[
g(x_1, \ldots, x_r) = H\left( \begin{array}{c}
1, \ldots, 1, 1 \\
x_r - x_{r-1}, \ldots, x_2 - x_1, x_1
\end{array} \right),
\]

or equivalently

\[
H\left( \begin{array}{c}
1, \ldots, 1 \\
y_1, \ldots, y_r
\end{array} \right) = g(y_r, y_{r-1} + y_r, \ldots, y_1 + \cdots + y_r).
\]
$q$-analogues of multiple zeta values: shuffle product

So if we multiply the generating series in depth one we get

\[
g(x) \cdot g(y) = H\left(\frac{1}{x}\right) \cdot H\left(\frac{1}{y}\right)
\]

\[
= H\left(\frac{1,1}{x,y}\right) + H\left(\frac{1,1}{y,x}\right) + H\left(\frac{2}{x_1 + x_2}\right)
\]

\[
= g(x, x + y) + g(y, x + y) + H\left(\frac{2}{x + y}\right).
\]
$q$-analogues of multiple zeta values: shuffle product

**MZV:** Recall that the generating series of the shuffle regularized multiple zeta values

$$\mathcal{T}^{\shuffle}(x_1, \ldots, x_r) = \sum_{k_1, \ldots, k_r \geq 1} \zeta^{\shuffle}(k_1, \ldots, k_r) x_1^{k_1-1} \ldots x_r^{k_r-1}$$

satisfy

$$\mathcal{T}^{\shuffle}(x) \cdot \mathcal{T}^{\shuffle}(y) = \mathcal{T}^{\shuffle}(x, x+y) + \mathcal{T}^{\shuffle}(y, x+y).$$

**qMZV:** The generating series $\mathcal{g}(x_1, \ldots, x_r)$ satisfies similar equations, e.g.

$$\mathcal{g}(x) \cdot \mathcal{g}(y) = \mathcal{g}(x, x+y) + \mathcal{g}(y, x+y) + H\left(\begin{array}{c} 2 \\ x+y \end{array}\right).$$
MZV: Recall that the generating series of the shuffle regularized multiple zeta values

\[ \mathcal{Z}^{\shuffle}(x_1, \ldots, x_r) = \sum_{k_1, \ldots, k_r \geq 1} \zeta^{\shuffle}(k_1, \ldots, k_r) x_1^{k_1-1} \cdots x_r^{k_r-1} \]

satisfy

\[ \mathcal{Z}^{\shuffle}(x) \cdot \mathcal{Z}^{\shuffle}(y) = \mathcal{Z}^{\shuffle}(x, x + y) + \mathcal{Z}^{\shuffle}(y, x + y). \]

qMZV: The generating series \( g(x_1, \ldots, x_r) \) satisfies similar equations, e.g.

\[ g(x) \cdot g(y) = g(x, x + y) + g(y, x + y) + H\left( \frac{2}{x+y} \right). \]

- The properties of the function \( H \) can be used to define series \( g^{\shuffle} \) satisfying the shuffle product formula of MZV.
- We now want to explain the connection of \( H\left( \frac{2}{x+y} \right) \) and the operator \( D = q \frac{d}{dq} \).
First notice that

\[
D \frac{q^d}{1 - q^d} = q \frac{d}{dq} \frac{q^d}{1 - q^d} = d \left( \frac{q^d}{1 - q^d} \right)^2 + d \frac{q^d}{1 - q^d},
\]

which leads to

\[
\sum_{k > 0} Dg_k(q)x^{k-1} = Dg(x) = DH \left( \frac{1}{x} \right) = D \sum_{0 < d} e^{dx} \frac{q^d}{1 - q^d}
\]

\[
= \sum_{0 < d} de^{dx} \left( \frac{q^d}{1 - q^d} \right)^2 + \sum_{0 < d} de^{dx} \frac{q^d}{1 - q^d}
\]

\[
= \frac{d}{dy} \left( H \left( \frac{2}{x + y} \right) + H \left( \frac{1}{x + y} \right) \right) \bigg|_{y=0}.
\]

\[
= g(x+y)
\]
With $H\left(\frac{2}{x+y}\right) = g(x) \cdot g(y) - g(x, x+y) - g(y, x+y)$ we get

$$\sum_{k>0} Dg_k(q)x^{k-1} = \frac{d}{dy}\left(g(x) \cdot g(y) - g(x, x+y) - g(y, x+y) + g(x+y)\right)|_{y=0}.$$

In particular this proves that $Dg_k(q) \in G$. 
With \( H\left(\frac{2}{x+y}\right) = g(x) \cdot g(y) - g(x, x + y) - g(y, x + y) \) we get

\[
\sum_{k>0} Dg_k(q)x^{k-1} = \frac{d}{dy}\left( g(x) \cdot g(y) - g(x, x + y) - g(y, x + y) + g(x + y) \right) \bigg|_{y=0}.
\]

In particular this proves that \( Dg_k(q) \in \mathcal{G} \).

Since \( \frac{d}{dy} g(y) \bigg|_{y=0} \) equals \( g_2(q) \) this can be interpreted as

\[
\text{Derivative of } g_k(q) = \text{Failure of the shuffle product formula for } g_k(q) \cdot g_2(q) + \text{Lower weight terms}
\]
Derivative of $g_k(q) = \text{Failure of the shuffle product formula for } g_k(q) \cdot g_2(q) + \text{Lower weight terms}$

Example

The shuffle product formula of $\zeta(3) \cdot \zeta(2)$ reads

$$\zeta(3) \cdot \zeta(2) = \zeta(3,2) + 3\zeta(2,3) + 6\zeta(1,4). \quad (1)$$

The derivative of $g_3(q)$ is given by

$$D g_3(q) = g_3(q) \cdot g_2(q) - g_{3,2}(q) - 3g_{2,3}(q) - 6g_{1,4}(q) + 3g_4(q). \quad (2)$$

Notice: Multiplying (2) by $(1 - q)^5$ and taking the limit $q \to 1$ one obtains (1).
This can be done for arbitrary depth:

**Theorem**

The derivative of the generating series $g$ can be written as

$$Dg(x_1, \ldots, x_r) = g(x_1, \ldots, x_r) \cdot g_2(q)$$

$$- \frac{d}{dy} \left( \sum_{j=0}^{r} g(x_1, x_2, \ldots, x_{r-j}, x_{r-j} + y, x_{r-j+1} + y, \ldots, x_r + y) \right) \bigg|_{y=0}$$

$$- \frac{d}{dy} \left( \sum_{j=1}^{r} g(x_1, \ldots, x_{j-1}, x_j + y, \ldots, x_r + y) \right) \bigg|_{y=0}$$

In particular the space $G$ is closed under $D$. 
This can be done for arbitrary depth:

**Theorem**

The derivative of the generating series $\mathcal{g}$ can be written as

$$
D \mathcal{g}(x_1, \ldots, x_r) = \mathcal{g}(x_1, \ldots, x_r) \cdot g_2(q) \\
- \frac{d}{dy} \left( \sum_{j=0}^{r} \mathcal{g}(x_1, x_2, \ldots, x_{r-j}, x_{r-j} + y, x_{r-j+1} + y, \ldots, x_r + y) \right) \bigg|_{y=0}
$$

$$
- \frac{d}{dy} \left( \sum_{j=1}^{r} \mathcal{g}(x_1, \ldots, x_{j-1}, x_j + y, \ldots, x_r + y) \right) \bigg|_{y=0}
$$

In particular the space $\mathcal{G}$ is closed under $D$.

Derivative of $g_{k_1,\ldots,k_r}(q) = \text{Failure of the shuffle product formula for } g_{k_1,\ldots,k_r}(q) \cdot g_2(q) + \text{Lower weight & depth terms}$
We now want to study certain subspaces of our $q$-analogues which we denote by

$$\mathcal{G}^{\geq 2} := \langle g_{k_1, \ldots, k_r}(q) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle \subset \mathcal{G},$$

$$\mathcal{Z}_q^{\geq 2} := \langle \zeta_q(k_1, \ldots, k_r) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle \subset \mathcal{Z}_q.$$

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^{\geq 2} = \mathcal{Z}_q^{\geq 2}$. 
We now want to study certain subspaces of our $q$-analogues which we denote by

$$
\mathcal{G}^{\geq 2} := \langle g_{k_1, \ldots, k_r}(q) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle_{\mathbb{Q}} \subset \mathcal{G},
$$

$$
\mathcal{Z}_{q}^{\geq 2} := \langle \bar{\zeta}_q(k_1, \ldots, k_r) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle_{\mathbb{Q}} \subset \mathcal{Z}_q.
$$

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^{\geq 2} = \mathcal{Z}_{q}^{\geq 2}$.

**Theorem**

Define for $2 \leq s \leq k$ the numbers $\alpha_{s,k} \in \mathbb{Q}$ by

$$
\sum_{s=2}^{k} \frac{\alpha_{s,k}}{(s-1)!} X^{s-1} := \left( \frac{X}{k-1} \right) = \frac{X(X-1)\ldots(X-k+2)}{(k-1)!}.
$$

Then we have for $k_1, \ldots, k_r \geq 2$

$$
\bar{\zeta}_q(k_1, \ldots, k_r) = \sum_{2 \leq s_j \leq k_j, 1 \leq j \leq r} \alpha_{s_1,k_1} \cdots \alpha_{s_r,k_r} g_{s_1,\ldots,s_r}(q).
$$
We now want to study certain subspaces of our $q$-analogues which we denote by

\[
\mathcal{G}^\geq_2 := \langle g_{k_1, \ldots, k_r}(q) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle_\mathbb{Q} \subset \mathcal{G},
\]

\[
\mathcal{Z}_q^\geq_2 := \langle \zeta_q(k_1, \ldots, k_r) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle_\mathbb{Q} \subset \mathcal{Z}_q.
\]

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^\geq_2 = \mathcal{Z}_q^\geq_2$.

**Theorem**

Define for $2 \leq s \leq k$ the numbers $\beta_{k,s} \in \mathbb{Q}$ by

\[
\sum_{2 \leq s \leq k < \infty} \beta_{k,s} T^{k-s} X^k = \frac{XT}{T + 1 - e^{XT}} - X.
\]

Then we have for $k_1, \ldots, k_r \geq 2$

\[
g_{k_1, \ldots, k_r}(q) = \sum_{2 \leq s_j \leq k_j \atop 1 \leq j \leq r} \beta_{s_1,k_1} \cdots \beta_{s_r,k_r} \zeta_q(s_1, \ldots, s_r).
\]
We now want to study certain subspaces of our $q$-analogues which we denote by

$$\mathcal{G}^{\geq 2} := \langle g_{k_1, \ldots, k_r}(q) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle_{\mathbb{Q}} \subset \mathcal{G},$$

$$\mathcal{Z}_q^{\geq 2} := \langle \zeta_q(k_1, \ldots, k_r) \mid r \geq 0, k_1, \ldots, k_r \geq 2 \rangle_{\mathbb{Q}} \subset \mathcal{Z}_q.$$

Even though $\mathcal{Z}_q \neq \mathcal{G}$ it is easy to prove that $\mathcal{G}^{\geq 2} = \mathcal{Z}_q^{\geq 2}$. 
We now want to study certain subspaces of our $q$-analogues which we denote by

$$G_{\geq 2} := \langle g_{k_1,\ldots,k_r}(q) \mid r \geq 0, k_1,\ldots,k_r \geq 2 \rangle_{\mathbb{Q}} \subset G,$$

$$Z_{q}^{\geq 2} := \langle \zeta_q(k_1,\ldots,k_r) \mid r \geq 0, k_1,\ldots,k_r \geq 2 \rangle_{\mathbb{Q}} \subset Z_q.$$

Even though $Z_q \neq G$ it is easy to prove that $G_{\geq 2} = Z_q^{\geq 2}$.

**Conjecture**

The space $G_{\geq 2}$ is close under $D = q \frac{d}{dq}$. 

**Diagram**

- $G$
- $q \frac{d}{dq}$
- $G_{\geq 2}$
- $q \frac{d}{dq}$
- $Z_q^{\geq 2}$
- quasi modular forms
- $q \frac{d}{dq}$
Motivation 1: Multiple Eisenstein series

For $k_1, \ldots, k_r \geq 2$ the multiple Eisenstein series $G_{k_1, \ldots, k_r}(\tau)$ is defined by

$$G_{k_1, \ldots, k_r}(\tau) = \sum_{0 \prec \lambda_1 \prec \cdots \prec \lambda_r, \lambda_i \in \mathbb{Z} \tau + \mathbb{Z}} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}},$$

where $\tau \in \{x + iy \in \mathbb{C} \mid y > 0\}$ is an element in the upper half plane and the order $\prec$ on $\mathbb{Z} \tau + \mathbb{Z}$ is defined by

$$m_1 \tau + n_1 \prec m_2 \tau + n_2 :\iff (m_1 < m_2) \lor (m_1 = m_2 \land n_1 < n_2).$$
Motivation 1: Multiple Eisenstein series

For \( k_1, \ldots, k_r \geq 2 \) the multiple Eisenstein series \( G_{k_1, \ldots, k_r}(\tau) \) is defined by

\[
G_{k_1, \ldots, k_r}(\tau) = \sum_{\lambda \in \mathbb{Z}^r, \lambda_1 \prec \cdots \prec \lambda_r} \frac{1}{\lambda_1^{k_1} \cdots \lambda_r^{k_r}},
\]

where \( \tau \in \{ x + iy \in \mathbb{C} \mid y > 0 \} \) is an element in the upper half plane and the order \( \prec \) on \( \mathbb{Z} \tau + \mathbb{Z} \) is defined by

\[
m_1 \tau + n_1 \prec m_2 \tau + n_2 :\iff (m_1 < m_2) \lor (m_1 = m_2 \land n_1 < n_2).
\]

Theorem

Setting \( q = \exp(2\pi i \tau) \) the \( \mathbb{C} \)-vector space spanned by all multiple Eisenstein series \( G_{k_1, \ldots, k_r}(\tau) \) with \( k_1, \ldots, k_r \geq 2 \) equals \( \mathbb{C} \otimes G^{\geq 2} \).

Conjecture

The \( \mathbb{C} \)-vector space spanned by all \( G_{k_1, \ldots, k_r}(\tau) \) with \( k_1, \ldots, k_r \geq 2 \) is closed under

\[
\frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq} = D.
\]
Motivation 2: Hilbert Scheme of surfaces

$q$-analogues of multiple zeta values also appear in algebraic geometry.

- $S$: nonsingular quasi-projective surface
- $\text{Hilb}(S, n)$: Hilbert scheme (parametrizes 0-dim. length $n$ subschemes of $S$)

In a recent work A. Okounkov introduces for a characteristic class $f$ on $S$ a $q$-series

$$\langle f \rangle = \sum_{n>0} \left( \int_{\text{Hilb}(S,n)} \ldots \right) q^n .$$

**Conjecture (Okounkov)**

For every characteristic class $f$ on $S$ it is $\langle f \rangle \in \mathcal{G}^{\geq 2}$.

Using geometric arguments one can show that for a certain characteristic class $c$ on $S$ it is

$$D \langle f \rangle = q \frac{d}{dq} \langle f \rangle = \langle f \cdot c \rangle - g_2(q) ,$$

which also lead Okounkov to the Conjecture that $\mathcal{G}^{\geq 2}$ is closed under $D$. 
We can not use the formula from the Theorem before, since we have for example

\[ Dg_3(q) = g_3(q) \cdot g_2(q) - g_{3,2}(q) - 3g_{2,3}(q) - 6g_{1,4}(q) + 3g_4(q). \]

All elements on the right side are in \( G^{\geq 2} \) except for \( g_{1,4}(q) \).
Derivatives in $G^\geq 2$

\[ Dg_3(q) = g_3(q) \cdot g_2(q) - g_{3,2}(q) - 3g_{2,3}(q) - 6g_{1,4}(q) + 3g_4(q). \]

All elements on the right side are in $G^\geq 2$ except for $g_{1,4}(q)$.

Above formula was obtained by considering the coefficient of $x^2$ in

\[
Dg(x) = \sum_{0<d} de^{dx} \left( \frac{q^d}{1-q^d} \right)^2 + \sum_{0<d} de^{dx} \frac{q^d}{1-q^d}
\]

\[
= \frac{d}{dy} \left( \begin{array}{c} 2 \\ x+y \end{array} \right) + \frac{1}{x+y} \Bigg|_{y=0}
\]

\[
= \frac{d}{dy} \left( g(x) \cdot g(y) - g(x, x+y) - g(y, x+y) + g(x + y) \right) \Bigg|_{y=0}.
\]
Derivatives in $G^{\geq 2}$

\[ D g_3(q) = 3g_1(q) \cdot g_4(q) - 6g_{1,4}(q) - 3g_{2,3}(q) - 3g_{3,2}(q) - 3g_{4,1}(q) + 3g_4(q). \]

All elements on the right side are in $G^{\geq 2}$ except for $g_{1,4}(q)$, $g_{4,1}(q)$ and $g_1(q) \cdot g_4(q)$.

Above formula was obtained by considering the coefficient of $x^2$ in

\[
D g(x) = \sum_{0<d} de^{dx} \left( \frac{q^d}{1 - q^d} \right)^2 + \sum_{0<d} de^{dx} \frac{q^d}{1 - q^d}
\]

\[ = \frac{d}{dx} \left( H \left( \frac{2}{x+y} \right) + H \left( \frac{1}{x+y} \right) \right) \bigg|_{y=0}
\]

\[ = \frac{d}{dx} \left( g(x) \cdot g(y) - g(x, x+y) - g(y, x+y) + g(x+y) \right) \bigg|_{y=0}. \]

Clearly the $\frac{d}{dy}$ can also be replaced by $\frac{d}{dx}$. 
Derivatives in $G_{\geq 2}$

$$
Dg_3(q) = 5g_5(q) - 4g_{2,3}(q) - 6g_{3,2}(q) + \frac{7}{12}g_3(q).
$$

All the elements on the right side are in $G_{\geq 2}$.

Above formula was obtained by considering the coefficient of $x^2$ in

$$
Dg(x) = \sum_{0<d} de^{dx} \left( \frac{q^d}{1-q^d} \right)^2 + \sum_{0<d} de^{dx} \frac{q^d}{1-q^d}
$$

$$
= \left( 2 \frac{d}{dx} - \frac{d}{dy} \right) \left( H \left( \frac{2}{x+y} \right) + H \left( \frac{1}{x+y} \right) \right) \bigg|_{y=0}
$$

$$
= \left( 2 \frac{d}{dx} - \frac{d}{dy} \right) \left( g(x) \cdot g(y) - g(x,x+y) - g(y,x+y) + g(x+y) \right) \bigg|_{y=0}.
$$

Instead of $\frac{d}{dy}$ and $\frac{d}{dx}$ we can also use $2 \frac{d}{dx} - \frac{d}{dy}$.

(and evaluate the product by using the harmonic product).
Derivatives in $G^{\geq 2}$: Depth one

**Theorem**

For $k \geq 1$ the derivative of $g_k(q)$ is given by

\[
Dg_k(q) = q \frac{d}{dq} g_k(q) = (2k - 1) g_{k+2}(q) - \sum_{j=2}^{k} (k + j - 1) g_{j,k+2-j}(q) - g_{k,2}(q)
\]

\[
+ \sum_{j=2}^{k} \frac{B_{k+2-j}}{(k + 2 - j)!} (3k - j + 1) g_j(q) + (-1)^k \frac{B_k}{k!} g_2(q).
\]

In particular $Dg_k(q) \in G^{\geq 2}$ for $k \geq 2$.

**Example:**

\[
q \frac{d}{dq} g_2(q) = 3 g_4(q) - 4 g_{2,2}(q) + \frac{1}{2} g_2(q).
\]

Notice that by multiplying both sides with $(1 - q)^{k+2}$ and taking the limit $q \to 1$ we obtain

\[
(2k - 1) \zeta(k + 2) = \sum_{j=2}^{k} (k + j - 1) \zeta(j, k + 2 - j) + \zeta(k, 2).
\]
From the Theorem we obtain inductively the following corollary

**Corollary**

For every \( k \geq 2 \) we have \( Dg_{k}, \ldots, k(q) \in G^{\geq 2} \).
Derivatives of $G^{\geq 2}$: Higher depths

From the Theorem we obtain inductively the following corollary

**Corollary**

For every $k \geq 2$ we have $Dg_{k,\ldots,k}(q) \in G^{\geq 2}$.

For even $k$ this can also proven without the Theorem by showing that

$$g_{k,\ldots,k}(q) \in \mathbb{Q}[\tilde{G}_2(q), \tilde{G}_4(q), \tilde{G}_6(q)].$$

**Theorem**

The series $g_{\{2\}^r}(q) = g_{2,\ldots,2}(q)$ is the coefficient of $X^{2r+1}$ in

$$2 \arcsin \left( \frac{X}{2} \right) \exp \left( \sum_{j \geq 1} \frac{(-1)^{j-1}}{j} \tilde{G}_{2j}(q) \left( 2 \arcsin \left( \frac{X}{2} \right) \right)^{2j} \right).$$

**Proof idea:** Use an explicit formula for the Fourier expansion of Multiple Eisenstein series.
We can also obtain the $\zeta_q$ version of our Theorem:

**Theorem**

For $k \geq 3$ the derivative of $\zeta_q(k)$ is given by

$$D\zeta_q(k) = (2k - 1)\zeta_q(k + 2) + 3(k - 1)\zeta_q(k + 1) + (k - 1)\zeta_q(k)$$

$$- \sum_{j=2}^{k-2} (k + j - 1) \left( \zeta_q(j, k + 2 - j) + \zeta_q(j + 1, k - j) \right)$$

$$- 2k\zeta_q(k, 2) - (2k - 2)\zeta_q(k - 1, 3) - k\zeta_q(2, k - 1)$$

and for $k = 2$ it is

$$D\zeta_q(2) = 3\zeta_q(4) + 3\zeta_q(3) + \zeta_q(2) - 4\zeta_q(2, 2).$$

In particular $D\zeta_q(k) \in \mathcal{Z}^\geq_q = \mathcal{G}^\geq_q$ for $k \geq 2$.

Notice that the depth one part is simpler but the depth two part is in weight $k + 2$ and $k + 1$. 

Derivatives of $\mathcal{G}^\geq_q$: Depth one for $\zeta_q$
Except for the example before and numerical experiments, there are no results (that I am aware of) in higher depths.

**Questions**

- Can we use a similar idea for higher depth by using our formula for $Dg(x_1, \ldots, x_r)$?
- Are there results on the derivatives in $\mathbb{Z}_q$ or $\mathbb{Z}_q^{\geq 2}$ for higher depths? (日本語で？)
- Is there another (better?) model to study the operator $D = q \frac{d}{dq}$?
In his work Okounkov also proposes a conjecture for the dimension of the associated graded algebra of $G_{\geq 2}$. For this let

$$G_{k}^{\geq 2} = \langle g_{k_1, \ldots, k_r}(q) \in G^{\geq 2} \mid r \geq 0, k_1 + \cdots + k_r = k \rangle_Q$$

and set $gr_0 G^{\geq 2} = \mathbb{Q}$ and for $k \geq 1$

$$gr_k G^{\geq 2} = G_k^{\geq 2} / G_{k-1}^{\geq 2}.$$

**Conjecture (Okounkov)**

The dimension $d_k = \dim_{\mathbb{Q}} gr_k G^{\geq 2}$ is given by

$$\sum_{k \geq 0} d_k x^k = \frac{1}{1 - x^2 - x^3 - x^4 - x^5 + x^8 + x^9 + x^{10} + x^{11} + x^{12}}.$$
ありがとうございます

Slides are available here: www.henrikbachmann.com