The double shuffle structure of certain q-analogues of multiple zeta values and their connections to modular forms

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Various Aspects of Multiple Zeta Values
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Slides are available here: www.henrikbachmann.com
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Multiple zeta values

**Definition**

For $k_1, \ldots, k_{r-1} \geq 1$, $k_r \geq 2$ define the multiple zeta value by

$$
\zeta(k_1, \ldots, k_r) = \sum_{0 < m_1 < \cdots < m_r} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.
$$

By $r$ we denote its depth and $k_1 + \cdots + k_r$ will be called its weight. For the $\mathbb{Q}$-vector space spanned by all multiple zeta values we write $\mathcal{Z}$.

- The product of two MZV can be expressed as a linear combination of MZV with the same weight (harmonic product). e.g:

  $$
  \zeta(k_1) \cdot \zeta(k_2) = \zeta(k_1, k_2) + \zeta(k_2, k_1) + \zeta(k_1 + k_2).
  $$

- MZV can be expressed as iterated integrals. This gives another way (shuffle product) to express the product of two MZV as a linear combination of MZV.

- These two products give a number of $\mathbb{Q}$-relations (double shuffle relations) between MZV.
Example:

\[ \zeta(3, 2) + 3\zeta(2, 3) + 6\zeta(1, 4) \overset{\text{shuffle}}{=} \zeta(2) \cdot \zeta(3) \overset{\text{harmonic}}{=} \zeta(2, 3) + \zeta(3, 2) + \zeta(5). \]

\[ \implies 2\zeta(2, 3) + 6\zeta(1, 4) \overset{\text{double shuffle}}{=} \zeta(5). \]

But there are more relations between MZV. e.g.:

\[ \zeta(1, 2) = \zeta(3). \]

These follow from the extended double shuffle relations.
Multiple Eisenstein series

- There are several connections of multiple zeta values to modular forms.
- One of them is given by multiple Eisenstein series $G_{k_1,\ldots,k_r}^{\boxplus}(\tau)$. In depth 1 these are the classical Eisenstein series

$$G_k^{\boxplus}(\tau) = \zeta(k) + \frac{(-2\pi i)^k}{(k-1)!} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n \quad (q = e^{2\pi i \tau}),$$

which are modular forms for even $k > 2$. ($\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$).
- These functions satisfy some of the double shuffle relations. For example it is

$$2G_{2,3}^{\boxplus}(\tau) + 6G_{1,4}^{\boxplus}(\tau) = G_5^{\boxplus}(\tau),$$

but

$$G_3^{\boxplus}(\tau) - G_{1,2}^{\boxplus}(\tau) = -\pi i \frac{d}{d\tau} G_1^{\boxplus}(\tau) \neq 0.$$
- There are a lot of open questions regarding multiple Eisenstein series.
Is the space spanned by all multiple Eisenstein series closed under \( \frac{d}{d\tau} = (2\pi i)q \frac{d}{dq} \)?

- The multiple Eisenstein series \( G_{k_1, \ldots, k_r}^{(\square)}(\tau) \) can be written as a \( \mathbb{Z}[2\pi i] \)-linear combination of certain \( q \)-series \( g_{k_1, \ldots, k_r}^{(\square)}(q) \in \mathbb{Q}[[q]] \). \( (\lambda = 2\pi i) \)

\[
G_{2,3}^{(\square)}(\tau) = \zeta(2, 3) + 3\zeta(3)\lambda^2 g_{2}^{(\square)}(q) + 2\zeta(2)\lambda^3 g_{3}^{(\square)}(q) + \lambda^5 g_{2,3}^{(\square)}(q).
\]

- The \( q \)-series \( g_{k_1, \ldots, k_r}^{(\square)}(q) \) can be written in terms of other \( q \)-series \( g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)}(q) \).

\[
g_{1,2}^{(\square)}(q) = g_{1,2}^{(0,0)}(q) + \frac{1}{2} g_{2}^{(1)}(q) - \frac{1}{2} g_{2}^{(0)}(q).
\]

- Most of the algebraic structure and the behavior of \( g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)}(q) \) under the operator \( q \frac{d}{dq} \) is well-understood.
Denote by $\mathcal{H} = \mathbb{Q}\langle e_0, e_1 \rangle$ the noncommutative polynomial algebra of indeterminates $e_0$ and $e_1$ over $\mathbb{Q}$.

Define its subalgebras $\mathcal{H}^0$ and $\mathcal{H}^1$ by

$$\mathcal{H}^0 = \mathbb{Q} \cdot 1 + e_1 \mathcal{H} e_0 \subset \mathcal{H}^1 = \mathbb{Q} \cdot 1 + e_1 \mathcal{H} \subset \mathcal{H}.$$ 

Set $e_k = e_1 e_0^{k-1}$ for $k \geq 1$.

The monomials $e_{k_1} \ldots e_{k_r}$ form a basis of $\mathcal{H}^1$.

The monomials $e_{k_1} \ldots e_{k_r}$ with $k_r \geq 2$ form a basis of $\mathcal{H}^0$. 
Define the \( \mathbb{Q} \)-bilinear commutative product \( \shuffle \) on \( \mathcal{H} \) for \( a, b \in \{e_0, e_1\} \) and \( v, w \in \mathcal{H} \) by

\[
1 \shuffle w = w \quad \shuffle 1 = w, \\
av \shuffle bw = a(v \shuffle bw) + b(\av \shuffle w).
\]

- The space \( \mathcal{H} \) equipped with this product becomes a commutative \( \mathbb{Q} \)-algebra which we denote by \( \mathcal{H}_\shuffle \).
- Both \( \mathcal{H}^1 \) and \( \mathcal{H}^0 \) are closed under \( \shuffle \) and by \( \mathcal{H}^1_\shuffle \) and \( \mathcal{H}^0_\shuffle \) we denote the corresponding subalgebras.
Harmonic (stuffle) product

Define the \(\mathbb{Q}\)-bilinear commutative product \(*\) on \(\mathcal{H}^1\) for \(k_1, k_2 \geq 1\) and \(v, w \in \mathcal{H}^1\) by

\[
1 \ast w = w \ast 1 = w,
\]

\[
e_{k_1} v \ast e_{k_2} w = e_{k_1} (v \ast e_{k_2} w) + e_{k_2} (e_{k_1} v \ast w) + e_{k_1+k_2}(v \ast w).
\]

- The space \(\mathcal{H}^1\) equipped with this product becomes a commutative \(\mathbb{Q}\)-algebra which we denote by \(\mathcal{H}_*\).
- The subspace \(\mathcal{H}^0\) is also closed under \(*\) and by \(\mathcal{H}_*^0\) we denote the corresponding subalgebra.
View $\zeta$ as a $\mathbb{Q}$-linear map from $\mathcal{H}^0$ to $\mathbb{Z} \subset \mathbb{R}$ which sends the monomials $e_{k_1} \ldots e_{k_r}$ to $\zeta(k_1, \ldots, k_r)$.

$\zeta$ is an algebra homomorphism from both $\mathcal{H}^0$ and $\mathcal{H}^0_*$ to $\mathbb{Z}$, i.e. for $w, v \in \mathcal{H}^0$

$$\zeta(w \shuffle v) = \zeta(w) \cdot \zeta(v) = \zeta(w \ast v).$$

The map $\zeta$ can be extended to algebra homomorphisms

$$\zeta^\shuffle: \mathcal{H}^1_\shuffle \rightarrow \mathbb{Z}$$

and

$$\zeta^*: \mathcal{H}^1_* \rightarrow \mathbb{Z},$$

which are uniquely determined by $\zeta^\shuffle(e_1) = \zeta^*(e_1) = 0$ and $\zeta^\shuffle(w) = \zeta^*(w) = \zeta(w)$ for $w \in \mathcal{H}^0$. 
Define for words $u, v \in \mathcal{H}^1$ the element $ds(u, v) \in \mathcal{H}^1$ by

$$ds(u, v) = u \ast v - u \shuffle v.$$  

If both $u, v \in \mathcal{H}^0$ we have $\zeta(ds(u, v)) = 0$.

But more generally we have the following Theorem, which conjecturally gives all linear relations between multiple zeta values.

**Theorem (Extended double shuffle relations)**

For $u \in \mathcal{H}^0$ and $v \in \mathcal{H}^1$ it is

$$\zeta^{\shuffle}(ds(u, v)) = \zeta^{\ast}(ds(u, v)) = 0.$$
Now we want to introduce a similar algebraic setup for our $q$-series $g_{k_1,\ldots,k_r}^{(d_1,\ldots,d_r)}(q)$.

For this consider the space $\mathcal{H}^2$ spanned by words in the double-indiced letters $e_k^{(d)}$ with $k \geq 1$ and $d \geq 0$, i.e. let

$$\mathcal{H}^2 = \mathbb{Q}\langle A \rangle$$

be the noncommutative polynomial algebra of indeterminates $A = \{e_k^{(d)} \mid k \geq 1, d \geq 0\}$ over $\mathbb{Q}$.

In the following we will define two products $\ast$ and $\cdot$ on $\mathcal{H}^2$. 


**Definition - The product** \( \star \) on \( H^2 \)

For \( w, v \in H^2 \), \( d_1, d_2 \geq 0 \) and \( k_1, k_2 \geq 1 \) define

\[
1 \star w = w \star 1 = w
\]

and

\[
e^{(d_1)} v \star e^{(d_2)} w = e^{(d_1)} (v \star e^{(d_2)} w) + e^{(d_2)} (e^{(d_1)} v \star w)
\]

\[
+ \binom{d_1 + d_2}{d_1} e^{(d_1+d_2)} (v \star w)
\]

\[
+ \binom{d_1 + d_2}{d_1} \sum_{j=1}^{k_1} \lambda_{k_1,k_2}^j e_j^{(d_1+d_2)} (v \star w)
\]

\[
+ \binom{d_1 + d_2}{d_1} \sum_{j=1}^{k_2} \lambda_{k_2,k_1}^j e_j^{(d_1+d_2)} (v \star w),
\]

where the numbers \( \lambda_{a,b}^j \in \mathbb{Q} \) for \( 1 \leq j \leq a \) are defined by

\[
\lambda_{a,b}^j := (-1)^{b-1} \binom{a + b - j - 1}{a - j} \frac{B_{a+b-j}}{(a + b - j)!}.
\]
Theorem

The space $\mathcal{H}^2$ equipped with the product $\boxtimes$ becomes a commutative $\mathbb{Q}$-algebra $\mathcal{H}^2$. For example we have

$$e_2^{(0)} \boxtimes e_3^{(0)} = e_2^{(0)} e_3^{(0)} + e_3^{(0)} e_2^{(0)} + e_5^{(0)} - \frac{1}{12} e_3^{(0)},$$

$$e_1^{(1)} \boxtimes e_1^{(2)} = e_1^{(1)} e_1^{(2)} + e_1^{(2)} e_1^{(1)} + 3e_2^{(3)} - 3e_1^{(3)}.$$

Notice that up to the term $-\frac{1}{12} e_3^{(0)}$ the first line looks exactly like the harmonic product

$$e_2 * e_3 = e_2 e_3 + e_3 e_2 + e_5$$

in $\mathcal{H}^1$. 
Recall: The product $\sqcup$ on $\mathcal{H}^1$ was defined by writing $e_k = e_1 e_0^{k-1}$ and using the shuffle product on $\mathbb{Q}\langle e_0, e_1 \rangle$.

For the second product $\boxdot$ on $\mathcal{H}^2$ we will use a different approach.

We will define an involution $P : \mathcal{H}^2 \to \mathcal{H}^2$ and then set for $u, v \in \mathcal{H}^2$

$$u \boxdot v = P(P(u) \boxdot P(v)).$$
Define the following element in $\mathcal{H}_2[[X_1, \ldots, X_r, Y_1, \ldots, Y_r]]$

$$M\left(\begin{array}{c} X_1, \ldots, X_r \\ Y_1, \ldots, Y_r \end{array}\right) := \sum_{k_1, \ldots, k_r \geq 1 \atop d_1, \ldots, d_r \geq 0} e_{k_1}^{(d_1)} \cdots e_{k_r}^{(d_r)} X_1^{k_1-1} \cdots X_r^{k_r-1} Y_1^{d_1} \cdots Y_r^{d_r}.$$ 

**Definition**

For $k_1, \ldots, k_r \geq 1$, $d_1, \ldots, d_l \geq 0$ and $w = e_{k_1}^{(d_1)} \cdots e_{k_r}^{(d_r)}$ define $P(w)$ as the coefficients of $X_1^{k_1-1} \cdots X_r^{k_r-1} Y_1^{d_1} \cdots Y_r^{d_r}$ in

$$M\left(\begin{array}{c} Y_r, Y_r-1 + Y_r, \ldots, Y_1 + \cdots + Y_r \\ X_r - X_{r-1}, X_{r-1} - X_{r-2}, \ldots, X_1 \end{array}\right).$$

Define the $\mathbb{Q}$-linear map $P : \mathcal{H}_2 \to \mathcal{H}_2$ by setting $P(1) = 1$ and extending the above definition on monomials linearly to $\mathcal{H}_2$.

Notice that the map $P$ is an involution on $\mathcal{H}_2$, i.e. $P(P(w)) = w$ for all $w \in \mathcal{H}_2$. 
For $r = 1$ the definition reads

$$\sum_{k_1 \geq 1 \atop d_1 \geq 0} P(e^{(d_1)}_{k_1}) X_1^{k_1-1} Y_1^{d_1} := M\left(\begin{array}{c} Y_1 \\ X_1 \end{array}\right) = \sum_{k_1 \geq 1 \atop d_1 \geq 0} e^{(d_1)}_{k_1} Y_1^{k_1-1} X_1^{d_1}$$

and therefore $P(e^{(d_1)}_{k_1}) = e^{(k_1-1)}_{d_1+1}$.

Other examples are

$$P(e^{(2)}_1 e^{(1)}_1) = e^{(0)}_2 e^{(0)}_3 + 3 e^{(0)}_1 e^{(0)}_4,$$

$$P(e^{(1)}_1 e^{(2)}_1) = e^{(0)}_3 e^{(0)}_2 + 2 e^{(0)}_2 e^{(0)}_3 + 3 e^{(0)}_1 e^{(0)}_4$$

which can be obtained by calculation the coefficient of $X_1^0 X_2^0 Y_1^2 Y_2^1$ (resp. $X_1^0 X_2^0 Y_1^1 Y_2^2$) in $M\left(\begin{array}{c} Y_2, Y_1+Y_2 \\ X_2-X_1, X_1 \end{array}\right)$. 
Definition - The product $\boxdot$ on $\mathfrak{H}^2$

Define on $\mathfrak{H}^2$ the product $\boxdot$ for $u, v \in \mathfrak{H}^2$ by

$$u \boxdot v = P(P(u) \boxast P(v)).$$

Theorem

The space $\mathfrak{H}^2$ equipped with the product $\boxdot$ becomes a commutative $\mathbb{Q}$-algebra $\mathfrak{H}^2_\boxdot$.

That this product is commutative and associative which follows from the fact that $P$ is an involution together with the properties of $\boxast$. 


Algebraic setup - $q$-analogue case - "shuffle product"

We have seen before that

\[
\begin{align*}
  e^{(1)}_1 \boxtimes e^{(2)}_1 &= e^{(1)}_1 e^{(2)}_1 + e^{(2)}_1 e^{(1)}_1 + 3 e^{(3)}_2 - 3 e^{(3)}_1, \\
  P(e^{(2)}_1 e^{(1)}_1) &= e^{(0)}_2 e^{(0)}_3 + 3 e^{(0)}_1 e^{(0)}_4, \\
  P(e^{(1)}_1 e^{(2)}_1) &= e^{(0)}_3 e^{(0)}_2 + 2 e^{(0)}_2 e^{(0)}_3 + 3 e^{(0)}_1 e^{(0)}_4
\end{align*}
\]

and \( P(e^{(d_1)}_{k_1}) = e^{(k_1-1)}_{d_1+1} \).

**Example**

The product \( e^{(0)}_2 \boxtimes e^{(0)}_3 \) in \( \mathcal{H}^2 \) is therefore given by

\[
\begin{align*}
  e^{(0)}_2 \boxtimes e^{(0)}_3 &= P(P(e^{(0)}_2) \boxtimes P(e^{(0)}_3)) = P(e^{(1)}_1 \boxtimes e^{(2)}_1) \\
  &= P(e^{(2)}_1 e^{(1)}_2 + e^{(2)}_1 e^{(1)}_1 + 3 e^{(3)}_2 - 3 e^{(3)}_1) \\
  &= e^{(0)}_3 e^{(0)}_2 + 3 e^{(0)}_2 e^{(0)}_3 + 6 e^{(0)}_1 e^{(0)}_4 + 3 e^{(1)}_4 - 3 e^{(0)}_4.
\end{align*}
\]

Compare this to the shuffle product \( e_2 \shuffle e_3 = e_3 e_2 + 3 e_2 e_3 + 6 e_1 e_4 \) on \( \mathcal{H}^1 \).
In analogy to $\zeta^{\square}$ and $\zeta^*$, which are algebra homomorphism from $\mathcal{H}_1^{\square}$ (resp. $\mathcal{H}_1^*$) to $\mathbb{R}$, we will now define a map

$$g : \mathcal{H}^2 \longrightarrow \mathbb{Q}[[q]]$$

which will be an algebra homomorphism from both $\mathcal{H}^2_{\square}$ and $\mathcal{H}^2_\Box$ to $\mathbb{Q}[[q]]$. 
Definition

For $k_1, \ldots, k_r \geq 1$, $d_1, \ldots, d_l \geq 0$ we define the following $q$-series in $\mathbb{Q}[[q]]$

$$g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)}(q) = \sum_{0<u_1<\cdots<u_r} \frac{u_1^{d_1}}{d_1!} \cdots \frac{u_r^{d_r}}{d_r!} \cdot \frac{v_1^{k_1-1} \cdots v_l^{k_r-1}}{(k_1-1)! \cdots (k_r-1)!} \cdot q^{u_1 v_1 + \cdots + u_r v_r}.$$ 

By $k_1 + \cdots + k_r + d_1 + \cdots + d_r$ we denote its weight and by $r$ its depth.

Since $q$ will be fixed the whole time we will also write $g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)}$ instead of $g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)}(q)$. 
q-analogues - the series $g_{k_1,\ldots,k_r}^{(d_1,\ldots,d_r)}$

**Definition**

For $k_1,\ldots,k_r \geq 1$, $d_1,\ldots,d_l \geq 0$ we define the following $q$-series in $\mathbb{Q}[[q]]$

$$g_{k_1,\ldots,k_r}^{(d_1,\ldots,d_r)}(q) := \sum_{0 < u_1 < \cdots < u_r \atop 0 < v_1, \ldots, v_r} \frac{u_1^{d_1}}{d_1!} \cdots \frac{u_r^{d_r}}{d_r!} \cdot \frac{v_1^{k_1-1} \cdots v_r^{k_r-1}}{(k_1-1)! \cdots (k_r-1)!} \cdot q^{u_1 v_1 + \cdots + u_r v_r}.$$ 

By $k_1 + \cdots + k_r + d_1 + \cdots + d_r$ we denote its weight and by $r$ its depth.

Since $q$ will be fixed the whole time we will also write $g_{k_1,\ldots,k_r}^{(d_1,\ldots,d_r)}$ instead of $g_{k_1,\ldots,k_r}^{(d_1,\ldots,d_r)}(q)$.

**Example:** In depth one we have

$$g_k^{(0)} = \sum_{0 < u_1 \atop 0 < v_1} \frac{v_1^{k-1}}{(k-1)!} q^{u_1 v_1} = \frac{1}{(k-1)!} \sum_{n>0} \sigma_{k-1}(n) q^n,$$

where $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$. 

For the $\mathbb{Q}$-vector space spanned by all of these $q$-series we write

$$\mathcal{G} := \left\langle g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)} \mid r \geq 0, k_1, \ldots, k_r \geq 1, d_1, \ldots, d_r \geq 0 \right\rangle_{\mathbb{Q}} ,$$

where we set $g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)} = 1$ for $r = 0$.

In the case $d_1 = \cdots = d_r = 0$ we write

$$g_{k_1, \ldots, k_r} := g_{k_1, \ldots, k_r}^{(0, \ldots, 0)}$$

and denote the subspace spanned by all of these by

$$\mathcal{G}^{(0)} := \left\langle g_{k_1, \ldots, k_r} \mid r \geq 0, k_1, \ldots, k_r \geq 1 \right\rangle_{\mathbb{Q}} \subset \mathcal{G}.$$
We now consider the following $\mathbb{Q}$-linear map from $\mathcal{H}^2$ to $\mathcal{G}$ which we define on the monomials by

$$g : \mathcal{H}^2 \longrightarrow \mathcal{G},$$

$$w = e_{k_1}^{(d_1)} \ldots e_{k_r}^{(d_r)} \longmapsto g(w) := g_{k_1,\ldots,k_r}^{(d_1,\ldots,d_r)}.$$

Set $g(1) = 1$ and extend it linearly to $\mathcal{H}^2$. 
q-analogues - the map $\mathfrak{g}$

**Theorem**

We have the following statements for the map $\mathfrak{g}$.

- The map $\mathfrak{g}$ is invariant under $P$, i.e. for all $w \in \mathcal{H}^2$ it is
  \[ \mathfrak{g}(P(w)) = \mathfrak{g}(w). \]

- $\mathfrak{g}$ is an algebra homorphism from $\mathcal{H}^2$ to $\mathbb{Q}[[q]]$ with respect to both products $\boxtimes$ and $\boxdot$, i.e. we have for all $u, v \in \mathcal{H}^2$
  \[ \mathfrak{g}(u \boxdot v) = \mathfrak{g}(u) \cdot \mathfrak{g}(v) = \mathfrak{g}(u \boxtimes v), \]
  where $\cdot$ denotes the usual multiplication of formal $q$-series in $\mathbb{Q}[[q]]$.

- In particular the space $\mathcal{G} = \mathfrak{g}(\mathcal{H}^2) \subset \mathbb{Q}[[q]]$ is an $\mathbb{Q}$-algebra.
The invariance under the map $P$ can be explained by interpreting the coefficients of the series $g_{k_1,\ldots,k_r}$ as a sum over partitions.

$$g(w) = \sum_{n>0} \left( \sum_{\mathcal{P} = n} f(\mathcal{P}) \right) q^n = \sum_{n>0} \left( \sum_{\mathcal{P} = n} f(\mathcal{P}) \right) q^n = g(P(w)).$$

The action of $P$ on the coefficient correspond to conjugating ($\mathcal{P} \to \mathcal{P}'$) the partitions.

For $g(u) \cdot g(v) = g(u \square v)$ we use explicit formulas for the generating functions of $g_{d_1,\ldots,d_r}$.

Finally $g(u \boxdot v) = g(u) \cdot g(v)$ follows from the two statement above, since

$$g(u \boxdot v) = g(P(P(u) \boxdot P(v))) = g(P(u) \boxdot P(v)) = g(P(u)) \cdot g(P(v)) = g(u) \cdot g(v).$$
The Theorem provides a large family of linear relations between the $q$-series $g_{k_1,\ldots,k_r}^{(d_1,\ldots,d_r)}$. We have seen before that

$$e_2^{(0)} \Box e_3^{(0)} = e_2^{(0)} e_3^{(0)} + e_3^{(0)} e_2^{(0)} + e_5^{(0)} - \frac{1}{12} e_3^{(0)},$$

$$e_2^{(0)} \Box e_3^{(0)} = e_2^{(0)} e_2^{(0)} + 3 e_2^{(0)} e_3^{(0)} + 6 e_1^{(0)} e_4^{(0)} + 3 e_4^{(1)} - 3 e_4^{(0)}$$

and therefore we obtain the relation

$$0 = g(e_2^{(0)} \Box e_3^{(0)}) - g(e_2^{(0)} \Box e_3^{(0)})$$

$$= g_5 - 2 g_{2,3} - 6 g_{1,4} - 3 g_{4}^{(1)} + 3 g_{4} - \frac{1}{12} g_{3}.$$
$\mathcal{H}^1$ and $\mathcal{H}^0$ have a natural embedding in $\mathcal{H}^2$, by sending a monomial $e_{k_1} \ldots e_{k_r}$ to $e_{k_1}^{(0)} \ldots e_{k_r}^{(0)}$. We view both $\mathcal{H}^1$ and $\mathcal{H}^0$ as subspaces of $\mathcal{H}^2$, i.e.

$$\mathcal{H}^0 \subset \mathcal{H}^1 \subset \mathcal{H}^2.$$ 

**Proposition**

The spaces $\mathcal{H}^1$ and $\mathcal{H}^0$ are closed under $\boxtimes$ and therefore we also have for $u, v \in \mathcal{H}^1$ (resp. $\mathcal{H}^0$) that

$$g(u) \cdot g(v) = g(u \boxtimes v).$$

In particular the space $\mathcal{G}^{(0)}$ is a subalgebra of $\mathcal{G}$.

Notice that the analogue statement for the product $\boxtimes$ is false, since $e_2 \boxtimes e_3 \not\in \mathcal{H}^1$.  

$q$-analogues - Subalgebra $\mathcal{G}^{(0)}$
Define for $k \in \mathbb{N}$ the map $Z_k : \mathbb{Q}[[q]] \to \mathbb{R} \cup \{\infty\}$ by

$$Z_k(f) = \lim_{q \to 1} (1 - q)^k f(q).$$

**Proposition**

If $k_r - d_r \geq 2$ and $k_j - d_j \geq 1$ for $j = 1, \ldots, r - 1$, then

$$Z_{k_1 + \cdots + k_r} \left( g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)} \right) = \frac{1}{d_1! \ldots d_r!} \zeta(k_1 - d_1, \ldots, k_r - d_r).$$

In particular for $k_r \geq 2$ we have

$$Z_k \left( g_{k_1, \ldots, k_r} \right) = \begin{cases} 
\zeta(k_1, \ldots, k_r), & k_1 + \cdots + k_r = k, \\
0, & k_1 + \cdots + k_r < k.
\end{cases}$$
We will now introduce the second $q$-series $g^{\Box}$, which appear in the Fourier expansion of Multiple Eisenstein series. For this we need the following series:

**Definition**

Define for $n_1, \ldots, n_r \geq 1$ the series

\[
H\left(\frac{n_1, \ldots, n_r}{x_1, \ldots, x_r}\right) = \sum_{0<d_1<\cdots<d_r} e^{d_1 x_1} \left(\frac{q^{d_1}}{1 - q^{d_1}}\right)^{n_1} \cdots e^{d_r x_r} \left(\frac{q^{d_r}}{1 - q^{d_r}}\right)^{n_r}.
\]

Notice that this series "satisfies" the harmonic product formula. For example:

\[
H\left(\frac{n_1}{x_1}\right) \cdot H\left(\frac{n_2}{x_2}\right) = H\left(\frac{n_1, n_2}{x_1, x_2}\right) + H\left(\frac{n_2, n_1}{x_2, x_1}\right) + H\left(\frac{n_1 + n_2}{x_1 + x_2}\right).
\]
For $k_1, \ldots, k_r \geq 1$ define the $q$-series $g_{k_1, \ldots, k_r}(q) \in \mathbb{Q}[[q]]$ as the coefficients of the following generating function:

$$g_{\sqcup}(x_1, \ldots, x_r) = \sum_{k_1, \ldots, k_r \geq 1} g_{k_1, \ldots, k_r}(q) x_1^{k_1-1} \cdots x_r^{k_r-1}$$

$$:= \sum_{r} \sum_{m=1}^{r} \frac{1}{i_1! \cdots i_m!} H \left( i_1, i_2, \ldots, i_m \right) H \left( x_r - x_{r-i_1}, x_{r-i_1} - x_{r-i_1-i_2}, \ldots, x_{i_m} \right).$$

Again we also write $g_{k_1, \ldots, k_r}$ instead of $g_{k_1, \ldots, k_r}(q)$.

Define the $\mathbb{Q}$-linear map $g_{\sqcup}$ from $\mathcal{H}^1$ to $\mathbb{Q}[[q]]$ on the monomials by

$$g_{\sqcup} : \mathcal{H}^1 \longrightarrow \mathbb{Q}[[q]],$$

$$w = e_{k_1} \cdots e_{k_r} \longmapsto g_{\sqcup}(w) := g_{k_1, \ldots, k_r}$$

and set $g_{\sqcup}(1) = 1$. 
Theorem

- For all $k_1, \ldots, k_r \geq 1$ we have $g_{k_1, \ldots, k_r} \in \mathcal{G}$.

- In the cases $k_1, \ldots, k_{r-1} \geq 2$, $k_r \geq 1$ it is $g_{k_1, \ldots, k_r} = g_{k_1, \ldots, k_r} \in \mathcal{G}^{(0)}$.

- The map $g^{\underline{\shuffle}}$ is an algebra homomorphism from $\mathcal{H}_{1_{\underline{\shuffle}}}^{1}$ to $\mathcal{G}$. 
q-analogues - $g^\uplus$

**Theorem**

- For all $k_1, \ldots, k_r \geq 1$ we have $g_{k_1, \ldots, k_r}^\uplus \in \mathcal{G}$.
- In the cases $k_1, \ldots, k_{r-1} \geq 2, k_r \geq 1$ it is $g_{k_1, \ldots, k_r}^\uplus = g_{k_1, \ldots, k_r} \in \mathcal{G}^{(0)}$.
- The map $g^\uplus$ is an algebra homomorphism from $\mathcal{H}^1_\uplus$ to $\mathcal{G}$.

**Proof ideas:**

- The first two statements follow again by using explicit expressions for the generating functions of the $q$-series $g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)}$.
- The third statement uses results (Exponentialmap) by Hoffman on quasi-shuffle products to turn the "harmonic" product of $H$ into the "shuffle" product.

We then use the general case of the following fact

$$g^\uplus(e_{k_1}) \cdot g^\uplus(e_{k_2}) = g^\uplus(e_{k_1 \uplus e_{k_2}}) \iff g^\sharp(x_1) \cdot g^\sharp(x_2) = g^\sharp(x_1, x_2) + g^\sharp(x_2, x_1),$$

where $g^\sharp(x_1, \ldots, x_r) = g_{\uplus}(x_1, x_1 + x_2, \ldots, x_1 + \cdots + x_r)$.
There are explicit formulas to write $g^{\square}$ in terms of $g$.

**Proposition**

- In depth two it is

  $$g_{k_1,k_2}^{\square} = g_{k_1,k_2} + \delta_{k_1,1} \cdot \frac{1}{2} \left( g_{k_2}^{(1)} - g_{k_2} \right)$$

- And in depth three it is

  $$g_{k_1,k_2,k_3}^{\square} = g_{k_1,k_2,k_3} + \delta_{k_1,1} \cdot \frac{1}{2} \left( g_{k_2,k_3}^{(1,0)} - g_{k_2,k_3} \right)$$
  
  $$+ \delta_{k_2,1} \cdot \frac{1}{2} \left( g_{k_1,k_3}^{(0,1)} - g_{k_1,k_3}^{(1,0)} - g_{k_1,k_3} \right)$$
  
  $$+ \delta_{k_1 \cdot k_2,1} \cdot \left( \frac{1}{6} g_{k_3}^{(2)} - \frac{1}{4} g_{k_3}^{(1)} + \frac{1}{6} g_{k_3} \right).$$
We will now focus on the action of the operator $d = \frac{d}{dq}$.

**Proposition**

For $k_1, \ldots, k_r \geq 1$, $d_1, \ldots, d_r \geq 0$ we have

$$d \, g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)} = \sum_{j=1}^{r} (d_j + 1) \cdot k_j \cdot g_{k_1, \ldots, k_j+1, \ldots, k_r}^{(d_1, \ldots, d_j+1, \ldots, d_r)}.$$ 

In particular the space $\mathcal{G}$ is closed under $d$. 
We will now focus on the action of the operator \( d = q \frac{d}{dq} \).

**Proposition**

For \( k_1, \ldots, k_r \geq 1, d_1, \ldots, d_r \geq 0 \) we have

\[
\frac{d}{dq} g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)} = \sum_{j=1}^{r} (d_j + 1) \cdot k_j \cdot g_{k_1, \ldots, k_j+1, \ldots, k_r}^{(d_1, \ldots, d_j+1, \ldots, d_r)}.
\]

In particular the space \( \mathcal{G} \) is closed under \( d \).

**Proof:** This is an easy consequence of the definition

\[
g_{k_1, \ldots, k_r}^{(d_1, \ldots, d_r)} := \sum_{0 < u_1 < \cdots < u_r} \sum_{0 < v_1, \ldots, v_r} \frac{u_1^{d_1}}{d_1!} \cdots \frac{u_r^{d_r}}{d_r!} \cdot \frac{v_1^{k_1-1} \cdots v_r^{k_r-1}}{(k_1-1)! \cdots (k_r-1)!} \cdot q^{u_1 v_1 + \cdots + u_r v_r}
\]

together with

\[
\frac{d}{dq} \sum_{n \geq 0} a_n q^n = \sum_{n > 0} n \cdot a_n q^n.
\]
Derivatives of $g$

Even though it is

$$d g_{k_1, k_2} = d g_{k_1, k_2}^{(0,0)} = k_1 g_{k_1+1, k_2}^{(1,0)} + k_2 g_{k_1, k_2+1}^{(0,1)},$$

we have the following result:

**Theorem**

The subalgebra $\mathcal{G}^{(0)} \subset \mathcal{G}$ is closed under the operator $d = q \frac{d}{dq}$.

The proof uses double shuffle relations for the functions $g$ to show that (for example)

$$k_1 g_{k_1+1, k_2}^{(1,0)} + k_2 g_{k_1, k_2+1}^{(0,1)} \in \mathcal{G}^{(0)}.$$
Derivatives of $g$

Even though it is

$$d g_{k_1, k_2} = d g^{(0,0)}_{k_1, k_2} = k_1 g^{(1,0)}_{k_1+1, k_2} + k_2 g^{(0,1)}_{k_1, k_2+1},$$

we have the following result:

**Theorem**

The subalgebra $G^{(0)} \subset G$ is closed under the operator $d = q \frac{d}{dq}$.

The proof uses double shuffle relations for the functions $g$ to show that (for example)

$$k_1 g^{(1,0)}_{k_1+1, k_2} + k_2 g^{(0,1)}_{k_1, k_2+1} \in G^{(0)}$$

More generally we expect the following strong statement:

**Conjecture**

It is $G^{(0)} = G$. 
Derivatives of $g^\sqcup$

Define for $k \geq 1$

$$G_{\leq k}^\sqcup := \langle g_{k_1,\ldots,k_r} \in G \mid k_1 + \cdots + k_r \leq k \rangle_{\mathbb{Q}},$$

We expect that $d G_{\leq k}^\sqcup \subset G_{\leq k+2}^\sqcup$, but so far we only know the following:

**Theorem**

For $k \geq 1$ and $d = q \frac{d}{dq}$ we have

$$\frac{1}{k} \, d \, g_k^\sqcup = (k + 1)g_{k+2}^\sqcup - \sum_{n=2}^{k+1} (2^n - 2)g_{k+2-n,n}^\sqcup.$$
For the space spanned by all multiple Eisenstein series we write

\[ E_k = \langle G_{k_1, \ldots, k_r}^{\cup} \mid k = k_1 + \cdots + k_r, k_1, \ldots, k_r \geq 1 \rangle_{\mathbb{Q}}. \]

**Expectation (Rough statement)**

We expect that the map \( F \), given by

\[
F : \mathcal{E}_k \longrightarrow G_{\leq k}^{\cup} / G_{\leq k-1}^{\cup} \\
G_{k_1, \ldots, k_r}^{\cup} \longmapsto g_{k_1, \ldots, k_r}^{\cup}
\]

is an isomorphism of \( \mathbb{Q} \)-algebras, which respects the action of \( d \).

Good thing: It is easy to obtain results in the space \( G_{\leq k}^{\cup} / G_{\leq k-1}^{\cup} \).
Proposition

For $k \geq 1$ we have

$$d g_k^{\square} = k \cdot g_{k+1}^{(1)} \equiv 2k \cdot g_k^{\square} (\text{ds}(e_1, e_{k+1})) \mod G_{\leq k+1}^{\square}$$

Proof: Notice that

$$g_{k_1,k_2}^{\square} = g_{k_1,k_2} + \delta_{k_1,1} \cdot \frac{1}{2} g_{k_2}^{(1)} \mod G_{\leq k_1+k_2-1}^{\square}$$

and

$$g_{k_1}^{\square} \cdot g_{k_2}^{\square} = g_{k_1} \cdot g_{k_2} = g_{k_1,k_2} + g_{k_2,k_1} + g_{k_1+k_2} \mod G_{\leq k_1+k_2-1}^{\square}.$$

With this we can "measure" the failure of the double shuffle relations for the series $g^{\square}$:

$$g^{\square} (\text{ds}(e_{k_1}, e_{k_2})) = g^{\square} (e_{k_1} \ast e_{k_2}) - g^{\square} (e_{k_1} \shuffle e_{k_2})$$

$$= g_{k_1,k_2}^{\square} + g_{k_2,k_1}^{\square} + g_{k_1+k_2}^{\square} - g_{k_1}^{\square} \cdot g_{k_2}^{\square}$$

$$\equiv \frac{1}{2} \delta_{k_1,1} g_{k_2}^{(1)} + \frac{1}{2} \delta_{k_2,1} g_{k_1}^{(1)} \mod G_{\leq k_1+k_2-1}^{\square}.$$
The analogue statement of

\[ \frac{d}{\mathfrak{m}} g_k^{(1)} = k \cdot g_{k+1}^{(1)} \equiv 2k \cdot g^{(1)}(\text{ds}(e_1, e_{k+1})) \mod G_{\leq k+1} \]

is also known for Eisenstein series:

**Theorem (Kaneko)**

For \( k \geq 1 \), the derivative of the Eisenstein series \( G_k^{(1)} \) can be written as

\[
(2\pi i)^2 \frac{d}{\mathfrak{m}} G_k^{(1)} = G_{1,k+1}^{(1)} + G_{k+1,1}^{(1)} + G_{k+2}^{(1)} - G_{k+1} \cdot G_1^{(1)} \\
= 2k \cdot G^{(1)}(\text{ds}(e_1, e_{k+1})) ,
\]

where in the last line we (by abuse of notation) interpret the multiple Eisenstein series as a map \( G^{(1)} : \mathcal{H}^1 \to \mathbb{C}[[q]] \).
Lemma

For \( k_1, k_2, k_3 \geq 1 \) and \( k = k_1 + k_2 + k_3 \) we have

\[
g^{\square}(ds(e_{k_1}, e_{k_2}e_{k_3})) \equiv \delta_{k_1,1}\frac{1}{2}g_{k_2,k_3}^{(0,1)} + \delta_{k_3,1}\frac{1}{2}\left(g_{k_2,k_1}^{(0,1)} - g_{k_2,k_1}^{(1,0)}\right) \\
+ \delta_{k_1}k_2,1\frac{1}{3}g_{k_3}^{(2)} + \delta_{k_2}k_3,1\frac{1}{6}g_{k_1}^{(2)} \mod G^{\square}_{\leq k-1}.
\]

Proposition

For \( k_1, k_2 \geq 2 \) we have

\[
d g_{k_1,k_2}^{\square} \equiv 2k_1\left(g_{k_1}^{\square}(ds(e_1, e_{k_1+1}e_{k_2})) - g_{k_2}^{\square}(ds(e_{k_2}, e_{k_1+1}e_1))\right) \\
+ 2k_2 \cdot g_{k_1}^{\square}(ds(e_1, e_{k_1}e_{k_2+1})) \mod G^{\square}_{\leq k_1+k_2+1}
\]

Proof: Use the lemma together with

\[
d g_{k_1,k_2}^{\square} = d g_{k_1,k_2} = k_1g_{k_1+1,k_2}^{(1,0)} + k_2g_{k_1,k_2+1}^{(0,1)}.
\]
Derivatives of q-analogues

We also have a similar result for $d g_{k_1, k_2, k_3}$, which leads to the following:

**Conjecture**

For $k_1, k_2, k_3 \geq 2$ the derivative of the Double and Triple Eisenstein series are given by

\[
(-2\pi i)^2 \, d \, G_{k_1, k_2} = 2k_1 \left( G_{\downarrow \uparrow} \left( ds(e_1, e_{k_1+1} e_{k_2}) \right) - G_{\downarrow \uparrow} \left( ds(e_{k_2}, e_{k_1+1} e_1) \right) \right) \\
+ 2k_2 \cdot G_{\downarrow \uparrow} \left( ds(e_1, e_{k_1} e_{k_2+1}) \right),
\]

\[
(-2\pi i)^2 \, d \, G_{k_1, k_2, k_3} = 2k_1 \cdot G_{\downarrow \uparrow} \left( ds(e_1, e_{k_1+1} e_{k_2} e_{k_3}) + ds(e_{k_3}, e_{k_2} e_{k_1+1} e_1) \right) \\
+ 2k_2 \cdot G_{\downarrow \uparrow} \left( ds(e_{k_3}, e_{k_1+1+k_2} e_1) - ds(e_{k_1+1} e_1, e_{k_2} e_{k_3}) \right) \\
+ 2k_2 \cdot G_{\downarrow \uparrow} \left( ds(e_1, e_{k_1} e_{k_2+1} e_{k_3}) - ds(e_{k_3}, e_{k_1} e_{k_2+1} e_1) \right) \\
+ 2k_3 \cdot G_{\downarrow \uparrow} \left( ds(e_1, e_{k_1} e_{k_2} e_{k_3+1}) \right)
\]

**Example:**

\[
d G_{2,2} \overset{?}{=} 8 G_{2,3,1} - 4 G_{1,2,3} + 4 G_{1,3,2} + 24 G_{1,4,1} - 4 G_{2,1,3} - 4 G_{2,2,2} \\
+ 4 G_{2,4} + 4 G_{3,3} + 4 G_{4,2} - 4 G_{5,1}.
\]