Tutorial 2: Vector spaces

A (real) **vector space** is a tuple $(V, +, \cdot)$, where V is a set together with two functions

$$\begin{aligned} + : V \times V &\longrightarrow V \\ (u, v) &\longmapsto u + v \end{aligned} \qquad\qquad \begin{aligned} \cdot : \mathbb{R} \times V &\longrightarrow V \\ (\lambda, v) &\longmapsto \lambda v \end{aligned}$$

such that the following properties are satisfied:

- Properties of the addition:

(A.1) $\forall u, v, w \in V\colon\ (u + v) + w = u + (v + w)$. (Associativity)

(A.2) $\forall u, v \in V\colon\ u + v = v + u$. (Commutativity)

(A.3) $\exists n \in V$, $\forall u \in V\colon\ n + u = u$. (Identity/neutral element of addition)

(A.4) $\forall u \in V$, $\exists v \in V\colon\ u + v = n$. (Inverse elements of addition)

- Compatibility of addition and scalar multiplication:

(C.1) $\forall u, v \in V$, $\lambda \in \mathbb{R}\colon\ \lambda \cdot (u + v) = \lambda u + \lambda v$. (Distributivity I)

(C.2) $\forall u \in V$, $\lambda, \mu \in \mathbb{R}\colon\ (\lambda + \mu) \cdot u = \lambda u + \mu u$. (Distributivity II)

(C.3) $\forall u \in V$, $\lambda, \mu \in \mathbb{R}\colon\ \lambda \cdot (\mu u) = (\lambda\mu) \cdot u$.

(C.4) $\forall u \in V\colon\ 1 \cdot u = u$.

Exercise 1. Let $V = \{x \in \mathbb{R} \mid x > 0\}$ and define for $u, v \in V$ and $\lambda \in \mathbb{R}$:

$$u \oplus v = uv,$$
$$\lambda \odot v = v^{\lambda}.$$

Show that $(V, \oplus, \odot)$ is a vector space.

- A polynomial functions is a function $f : \mathbb{R} \to \mathbb{R}$, such that there exist fixed $a_0, a_1, \dots, a_m \in \mathbb{R}$ with $f(x) = \sum_{j=0}^{m} a_j x^j$ for all $x \in \mathbb{R}$. The largest j with $a_j \neq 0$ is called the degree of f, denoted by $\deg(f)$.
- We denote the vector space of all polynomial functions by

$$\mathcal{P} = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a polynomial function}\},$$

where the addition and scalar multiplication is the usual one given on functions $\mathbb{R} \to \mathbb{R}$.

- For $n \geq 0$ denote by $\mathcal{P}_n = \{f \in \mathcal{P} \mid \deg(f) \leq n\}$ the space of polynomial functions of degree $\leq n$.

For example, the function $f(x) = x^3 + 2x$ is an element in $\mathcal{P}_m$ for all $m \geq 3$, but not in $\mathcal{P}_2$,$\mathcal{P}_1$ or $\mathcal{P}_0$.

Exercise 2. Consider the following subset of $\mathcal{P}_2$

$$U = \{f \in \mathcal{P}_2 \mid f(1) = 0\}.$$

Find a basis of U.

Tut Exercise 1: Need to check these:

- *Properties of the addition:*

(*A.1*) $\forall u, v, w \in V$: $(u \oplus v) \oplus w = u \oplus (v \oplus w)$. *(Associativity)*

(*A.2*) $\forall u, v \in V$: $u \oplus v = v \oplus u$. *(Commutativity)*

(*A.3*) $\exists n \in V, \forall u \in V$: $n \oplus u = u$. *(Identity/neutral element of addition)*

(*A.4*) $\forall u \in V, \exists v \in V$: $u \oplus v = n$. *(Inverse elements of addition)*

- *Compatibility of addition and scalar multiplication:*

(*C.1*) $\forall u, v \in V, \lambda \in \mathbb{R}$: $\lambda \odot (u \oplus v) = \lambda u \oplus \lambda v$. *(Distributivity I)*

(*C.2*) $\forall u \in V, \lambda, \mu \in \mathbb{R}$: $(\lambda + \mu) \odot u = \lambda u \oplus \mu u$. *(Distributivity II)*

(*C.3*) $\forall u \in V, \lambda, \mu \in \mathbb{R}$: $\lambda \odot (\mu u) = (\lambda\mu) \odot u$.

(*C.4*) $\forall u \in V$: $1 \cdot u = u$.

$u \oplus v := uv$

$\lambda \odot v := v^\lambda$

(A.1) $(u \oplus v) \oplus w$

$= (uv) \oplus w = (uv)w$

$= u(vw) = u \oplus (vw) = u \oplus (v \oplus w)$.

(A.2) clear

(A.3) choose $n = 1$, then $\forall u \in V$: $n \oplus u = 1 \cdot u = u$

(A.4) If $u \in V$ we can choose $v = \frac{1}{u}$ (since $u > 0$).

then $u \oplus v = u \cdot v = u \cdot \frac{1}{u} = 1 = n$.

(C.1): $\lambda \odot (u \oplus v) = \lambda \odot (uv) = (uv)^\lambda = u^\lambda v^\lambda = u^\lambda \oplus v^\lambda$

$= (\lambda \odot u) \oplus (\lambda \odot v)$

(C.2): $(\lambda+\mu)\odot u = u^{\lambda+\mu} = u^{\lambda} u^{\mu} = u^{\lambda} \oplus u^{\mu}$

$= \lambda \odot u \oplus \mu \odot u$

(C.3): $\lambda \odot (\mu \odot u) = \lambda \odot u^{\mu} = (u^{\mu})^{\lambda} = u^{\mu\lambda}$

$= (\mu\lambda) \odot u = (\lambda\mu) \odot u$

(C.4) $1 \odot u = u^{1} = u$. $\left((-1)\odot u = u^{-1} = \frac{1}{u} = \text{"}-u\text{"} \right)$

<u>Tut Exercise 2:</u>

$U = \{ f \in \mathbb{P}_2 \mid f(1) = 0 \} \subset \mathbb{P}_2$

Let $f \in \mathbb{P}_2$, then $f(x) = ax^2 + bx + c$ for some $a, b, c \in \mathbb{R}$.

If $f \in U$, then $f(1) = 0$, i.e. $a + b + c = 0$.

linear system!

We have two free variables,

Set $b = t_1$ and $c = t_2$ with $t_1, t_2 \in \mathbb{R}$. Then $a = -t_1 - t_2$

i.e we have $f(x) = (-t_1 - t_2)x^2 + t_1 x + t_2$

$= (-x^2 + x)t_1 + (-x^2 + 1)t_2$.

$\Rightarrow \quad U = \operatorname{span}\{ \underbrace{-x^2 + x}_{f_1(x)}, \underbrace{-x^2 + 1}_{f_2(x)} \}$

Claim: f_1 and f_2 are lin. indep. and form a basis of U.

Assume $\lambda_1 f_1 + \lambda_2 f_2 = 0$, i.e

$\forall x \in \mathbb{R} \quad \lambda_1 f_1(x) + \lambda_2 f_2(x) = 0$

Choose some values for x to show that $\lambda_1 = \lambda_2 = 0$:

$$\left.\begin{array}{ll} x=0 & \lambda_1 f_1(0) + \lambda_2 f_2(0) = \lambda_2 = 0 \\ x=-1 & \lambda_1 f_1(-1) + \lambda_2 f_2(-1) = -2\lambda_1 = 0 \end{array}\right\} \lambda_1 = \lambda_2 = 0$$

$\Rightarrow$ (f_1, f_2) is a basis of U.

Claim: we can set $f_3(x) = 1$ and get a basis (f_1, f_2, f_3) for $\mathcal{P}_2$

Homework 1: Vector spaces

Deadline: 22nd April (23:55 JST), 2024

Exercise 0. (2 Points)

(i) Try to solve the exercises below and write the solutions down by hand (paper, tablet) or by computer (Latex only). Create **one pdf-file** which contains your name on the first page and submit it before the deadline ends in TACT at the Assignment "Homework 1". Use precisely the following format as a filename: "**Familyname_Givenname_LA2_HW1.pdf**". Repeat this for future Homework by replacing HW1 with HW2, HW3, etc.. Points will be removed in future homeworks if this is not the case.

(ii) Read Chapter 14 of the lecture notes and compare the results and definitions with the corresponding results in Linear Algebra I (Chapters 1-13).

(You don't need to write down anything for Exercise 0)

Exercise 1. (3+2+2+1 = 8 Points) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an injective function. Define on $V := \mathrm{im}(\varphi)$ the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$u \oplus v = \varphi(\varphi^{-1}(u) + \varphi^{-1}(v)),$$
$$\lambda \odot v = \varphi(\lambda \cdot \varphi^{-1}(v)).$$

Here $+$ and $\cdot$ denote the usual addition and multiplication in $\mathbb{R}$.

(i) Show that $(V, \oplus, \odot)$ is a vector space. What is the neutral element of $(V, \oplus, \odot)$?
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

(ii) Determine all subspaces of $(V, \oplus, \odot)$.

(iii) Find an isomorphism

$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot).$$

Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

(iv) Do (ii) and (iii) explicitly for the case $\varphi(x) = e^x$.

Exercise 2. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\mathcal{P}_3 = \{f \in \mathcal{P} \mid \deg(f) \leq 3\},$$
$$U = \{f \in \mathcal{P}_3 \mid f(-2) = f(0) = 0\} \subset \mathcal{P}_3.$$

(i) Show that U is a subspace of $\mathcal{P}_3$.

(ii) Determine a basis $B = (b_1, \ldots, b_n)$ of U.

(iii) Determine the coordinate vector $[f]_B$ for the function $f \in U$ given by $f(x) = x(x+2)^2$.

(iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

Exercise 3. (2+2+2 = 6 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \left\{A \in \mathbb{R}^{2\times 2} \mid AM = MA\right\}.$$

(i) Show that for a given fixed $M \in \mathbb{R}^{2\times 2}$ the set $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$.

(ii) For $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

(iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have

$$2 \leq \dim(C(M)) \leq 4.$$

(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)

Hints for HW 1:

Ex 1 similar to tut Ex 1.
Ex 2 ——— " ——— Ex 2

Ex 3 : i) Check $0 \in C(M)$ (clear)

If $A, B \in C(M)$: $(A+B)M = \overset{\text{check}}{\ldots\ldots} = M(A+B)$
$(\lambda A)M = \ldots = M(\lambda A)$

ii) Assume $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. What does $SA = AS$ imply for a, b, c, d?

iii) $\mathbb{R}^{2\times 2}$ has basis $\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$

$\Rightarrow \dim \mathbb{R}^{2\times 2} = 4$

$$C(I_2) = \{ A \in \mathbb{R}^{2\times 2} \mid \overset{\text{always true}}{A I_2 = I_2 A} \}$$

$$= \mathbb{R}^{2\times 2}$$

$$\dim C(I_2) = 4$$

To show $\dim(C(M)) \geq 2$: Find matrices which are always in $C(M)$