Linear Algebra II — Spring 2024

Tutorial 1 — 11th April 2024

Review LA I:

- Linear systems $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 - x_2 = 5 \end{cases}$

- Matrices & Vectors $\underbrace{\begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} 3 \\ 5 \end{pmatrix}}_{b}$

$$(A|b) = \begin{pmatrix} 1 & 1 & | & 3 \\ 2 & -1 & | & 3 \end{pmatrix} \underset{-2}{\sim} \begin{pmatrix} 1 & 1 & | & 3 \\ 0 & -3 & | & -3 \end{pmatrix} \underset{-\frac{1}{3}}{\sim} \begin{pmatrix} 1 & 1 & | & 3 \\ 0 & 1 & | & 1 \end{pmatrix} \underset{-1}{\sim} \begin{pmatrix} 1 & 0 & | & 2 \\ 0 & 1 & | & 1 \end{pmatrix} = \operatorname{rref}(A|b)$$

row-reduced echelon form

Solution: $x_1 = 2$, $x_2 = 1$

$\mathbb{R}^{n\times 1} =: \mathbb{R}^n$: set of all vectors $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$\mathbb{R}^{m\times n}$: $m\times n$ matrice $m\Big\{ \overbrace{\begin{pmatrix} & & \\ & & \end{pmatrix}}^{n}$

For $x, y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ we defined

Addition: $x + y \in \mathbb{R}$

Scalar multiplication: $\lambda x \in \mathbb{R}$

- Linear maps: $F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$\forall x, y \in \mathbb{R}^n$ i) $F(x+y) = F(x) + F(y)$

$\forall \lambda \in \mathbb{R}$ ii) $F(\lambda x) = \lambda F(x)$

Theorem: If F is a lin. map. then $F(x) = [F]x$

$[F] \in \mathbb{R}^{m\times n}$ Matrix of F

- Subspaces $U \subset \mathbb{R}^n$ $\quad 0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$

i) $0 \in U$

ii) $x, y \in U \Rightarrow x+y \in U$

iii) $x \in U \Rightarrow \lambda x \in U$

Example: lines, planes, $\mathbb{R}^n$, $\{0\}$

- Image & Kernel of linear maps

$$F: \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$\{x \in \mathbb{R}^n \mid F(x) = 0\} = \ker(F) \subset \mathbb{R}^n$ $\qquad$ $\operatorname{im}(F) = \{y \in \mathbb{R}^m \mid \exists x \in \mathbb{R}^n : F(x) = y\} \subset \mathbb{R}^m$

Fact: Every subspace is the kernel and image of some linear map.

Example: $U = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \right\} = \ker(F)$

$$F: \mathbb{R}^2 \to \mathbb{R}, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto x_1 + x_2$$

- Linear independency & Bases

$v_1, \dots, v_\ell$ are lin. indep. if $\sum_{i=1}^{\ell} \lambda_i v_i = 0 \Rightarrow \lambda_i = 0 \ \forall i$.

$B = (b_1, \dots, b_m)$ is a basis of $U \subset \mathbb{R}^n$ if

1) $U = \operatorname{span}\{b_1, \dots, b_m\} = \{\sum_{i=1}^{m} \lambda_i b_i \mid \lambda_1, \dots, \lambda_m \in \mathbb{R}\}$

2) $b_1, \dots, b_m$ are lin. indep.

If $(b_1, \dots, b_m)$ is a basis of U then $\dim U = m$.

Linear Algebra II: Generalize all the above concepts in $\mathbb{R}^n$ to general <u>vector spaces</u>.

These are spaces which also have a notion of "+" (addition) and "·" scalar multiplication such that the "usual rules" are satisfied.

Example: $\mathcal{F}(\mathbb{R}, \mathbb{R})$: All functions $\mathbb{R} \to \mathbb{R}$

If $f, g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ we can define $f+g, \lambda \cdot f \in \mathcal{F}(\mathbb{R}, \mathbb{R})$

$$(f+g)(x) = f(x) + g(x)$$
$$(\lambda f)(x) = \lambda \cdot f(x)$$

$$\mathcal{F}(\mathbb{R},\mathbb{R})$$

$$\cup$$

$n \geq 0$

$$C^n(\mathbb{R},\mathbb{R}) = \{ f \in \mathcal{F}(\mathbb{R},\mathbb{R}) \mid f^{(n)} \text{ exists and is continuous} \}$$

$$\cup$$

$$C^0(\mathbb{R},\mathbb{R}) = \text{continuous functions } \mathbb{R} \to \mathbb{R}$$

$$\cup$$

$$\mathcal{P} = \text{polynomial functions } \mathbb{R} \to \mathbb{R}$$

$$\shortparallel$$

$$\cup \qquad \{ f \mid f(x) = \sum_{j=1}^{\ell} a_j x^j \text{ for some } \ell \geq 0 \text{ and } a_1, \ldots, a_\ell \in \mathbb{R} \}$$

$$\mathcal{P}_n = \text{polynomial functions of degree} \leq n$$

$$= \{ f \mid f(x) = \sum_{j=1}^{n} a_j x^j \text{ for some } a_1, \ldots, a_n \in \mathbb{R} \}$$

Homework 1: Vector spaces

Deadline: 22nd April (23:55 JST), 2024

Exercise 0. (2 Points)

(i) Try to solve the exercises below and write the solutions down by hand (paper, tablet) or by computer (Latex only). Create **one pdf-file** which contains your name on the first page and submit it before the deadline ends in TACT at the Assignment "Homework 1". Use precisely the following format as a filename: "**Familyname_Givenname_LA2_HW1.pdf**". Repeat this for future Homework by replacing HW1 with HW2, HW3, etc.. Points will be removed in future homeworks if this is not the case.

(ii) Read Chapter 14 of the lecture notes and compare the results and definitions with the corresponding results in Linear Algebra I (Chapters 1-13).

(You don't need to write down anything for Exercise 0)

Exercise 1. (3+2+2+1 = 8 Points) Let $\varphi : \mathbb{R} \to \mathbb{R}$ be an injective function. Define on $V := \operatorname{im}(\varphi)$ the addition $\oplus$ and the scalar multiplication $\odot$ for $u, v \in V$ and $\lambda \in \mathbb{R}$ by

$$\begin{aligned} u \oplus v &= \varphi(\varphi^{-1}(u) + \varphi^{-1}(v))\,, \\ \lambda \odot v &= \varphi(\lambda \cdot \varphi^{-1}(v)). \end{aligned}$$

Here $+$ and $\cdot$ denote the usual addition and multiplication in $\mathbb{R}$.

(i) Show that $(V, \oplus, \odot)$ is a vector space. What is the neutral element of $(V, \oplus, \odot)$?
(i.e. check that the operations $\oplus$ and $\odot$ satisfy the properties $(A.1) - (A.4)$ and $(C.1) - (C.4)$.)

(ii) Determine all subspaces of $(V, \oplus, \odot)$.

(iii) Find an isomorphism
$$F : (\mathbb{R}, +, \cdot) \longrightarrow (V, \oplus, \odot)\,.$$
Here $(\mathbb{R}, +, \cdot)$ denotes the vector space $\mathbb{R}^1$ with the usual addition and multiplication of real numbers.

(iv) Do (ii) and (iii) explicitly for the case $\varphi(x) = e^x$.

Exercise 2. (2+2+2+2 = 8 Points) Let $\mathcal{P}$ denote the set of all polynomial functions from $\mathbb{R}$ to $\mathbb{R}$. Define the following subsets

$$\begin{aligned} \mathcal{P}_3 &= \{f \in \mathcal{P} \mid \deg(f) \leq 3\}\,, \\ U &= \{f \in \mathcal{P}_3 \mid f(-2) = f(0) = 0\} \subset \mathcal{P}_3\,. \end{aligned}$$

(i) Show that U is a subspace of $\mathcal{P}_3$.

(ii) Determine a basis $B = (b_1, \dots, b_n)$ of U.

(iii) Determine the coordinate vector $[f]_B$ for the function $f \in U$ given by $f(x) = x(x+2)^2$.

(iv) Extend the basis B to a basis $\tilde{B}$ of $\mathcal{P}_3$.
(i.e. find a basis of $\mathcal{P}_3$, which contains all the basis elements of your basis B of U)

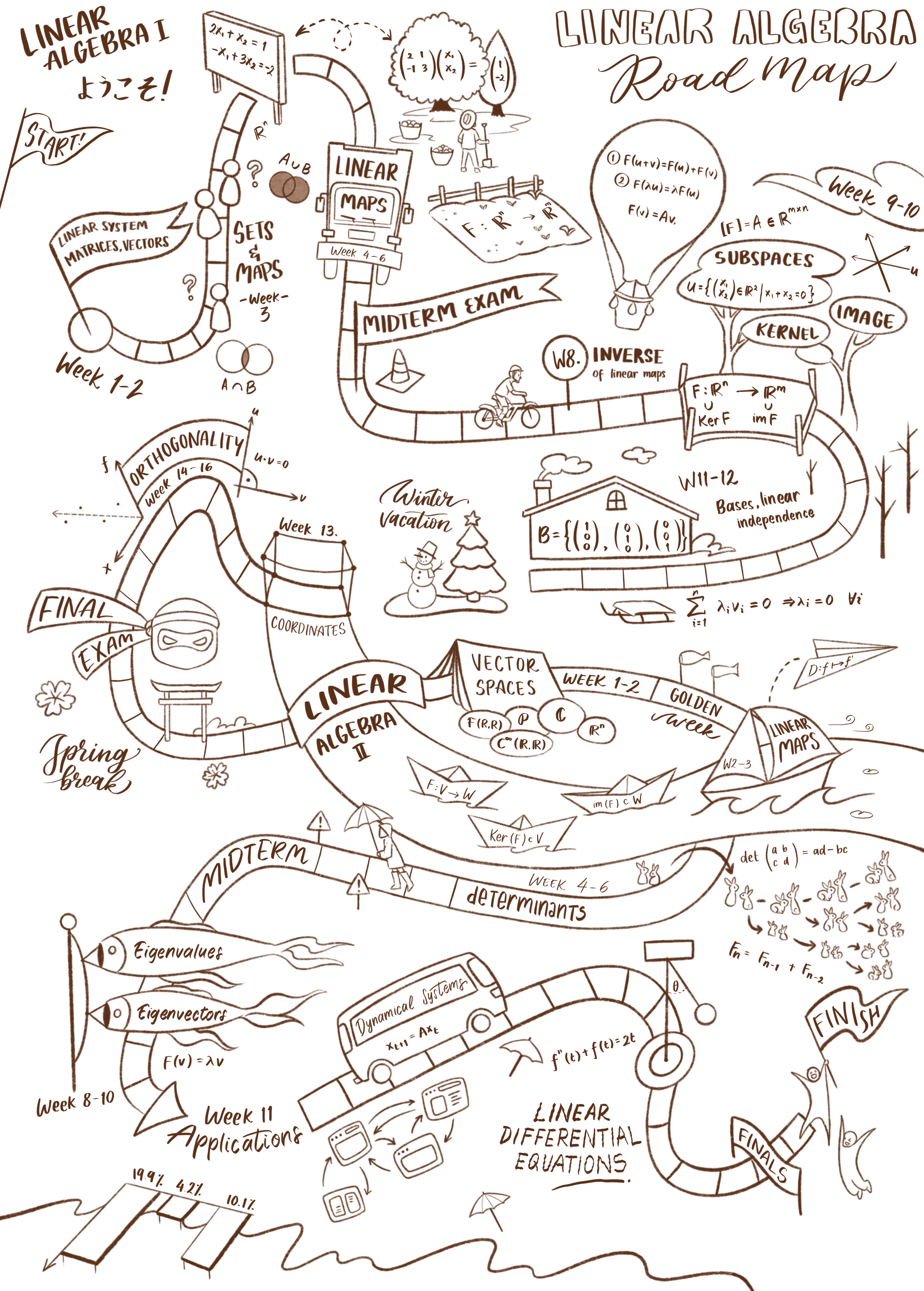
Exercise 3. (2+2+2 = 6 Points) Define for $M \in \mathbb{R}^{2\times 2}$ the following set

$$C(M) = \left\{A \in \mathbb{R}^{2\times 2} \mid AM = MA\right\}\,.$$

(i) Show that for a given fixed $M \in \mathbb{R}^{2\times 2}$ the set $C(M)$ is a subspace of $\mathbb{R}^{2\times 2}$.

(ii) For $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ determine a basis of $C(S)$.

(iii) Show that for all $M \in \mathbb{R}^{2\times 2}$ we have
$$2 \leq \dim(C(M)) \leq 4\,.$$
(i.e. show that there exists no matrix M, such that $C(M)$ has dimension 0 or 1.)


LINEAR ALGEBRA Road Map
LINEAR ALGEBRA I
ようこそ!
START!
$2x_1 + x_2 = 1$
$-x_1 + 3x_2 = -2$
$\begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$
$\mathbb{R}^n$
$A \cup B$
LINEAR SYSTEM MATRICES, VECTORS
SETS & MAPS -Week- 3
Week 1-2
$A \cap B$
LINEAR MAPS
Week 4-6
$F : \mathbb{R}^n \to \mathbb{R}^m$
① $F(u+v) = F(u) + F(v)$
② $F(\lambda u) = \lambda F(u)$
$F(v) = Av$
Week 9-10
$[F] = A \in \mathbb{R}^{m \times n}$
SUBSPACES
$U = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \mid x_1 + x_2 = 0 \}$
KERNEL
IMAGE
MIDTERM EXAM
W8. INVERSE of linear maps
$F : \mathbb{R}^n \to \mathbb{R}^m$
$\cup$ Ker F $\cup$ im F
W11-12 Bases, linear independence
$B = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \}$
$\sum_{i=1}^{n} \lambda_i v_i = 0 \Rightarrow \lambda_i = 0 \ \forall i$
ORTHOGONALITY
week 14-16
$u \cdot v = 0$
Week 13.
COORDINATES
Winter vacation
FINAL EXAM
LINEAR ALGEBRA II
VECTOR SPACES
WEEK 1-2
GOLDEN week
$F(\mathbb{R},\mathbb{R})$ $\mathcal{P}$ $\mathbb{C}$ $\mathbb{R}^n$ $C^\infty(\mathbb{R},\mathbb{R})$
$D : f \mapsto f'$
LINEAR MAPS
W2-3
$F : V \to W$
$\mathrm{im}(F) \subset W$
$\mathrm{Ker}(F) \subset V$
Spring break
MIDTERM
WEEK 4-6
determinants
$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$
$F_n = F_{n-1} + F_{n-2}$
Eigenvalues
Eigenvectors
$F(v) = \lambda v$
Week 8-10
Dynamical Systems
$x_{t+1} = Ax_t$
$f''(t) + f(t) = 2t$
θ
Week 11 Applications
19.9% 4.2% 10.1%
LINEAR DIFFERENTIAL EQUATIONS
FINISH
FINALS